EE611 Solutions to Problem Set 1

1. The following four waveforms are given:

$$\begin{aligned} s_0(t) &= rect(t) + rect(t-2) \\ s_1(t) &= rect(t-1) + rect(t-3) \\ s_2(t) &= rect(t-1) + rect(t-2) \\ s_3(t) &= rect(t-1) - rect(t-3) \end{aligned}$$

(i) (a) Determination of orthonormal basis functions usimg Gram-Schmidt procedure starting with $s_0(t)$ and going in sequence:
For $i = 0, 1, 2, 3$

$$g_i(t) = s_i(t) - \sum_{0}^{i-1} c_{ik} f_k(t),$$

$$f_i(t) = \frac{g_i(t)}{\sqrt{E_{g_i}}},$$

$$c_{ik} = \int_{-\infty}^{\infty} s_i(t) f_k(t) dt,$$

and

$$E_{g_i} = \int g_i^2(t) dt.$$

Initialization: $g_0(t) = s_0(t)$

$$\begin{aligned} f_0(t) &= \frac{s_0(t)}{\sqrt{E_{s_0}}} \\ E_{s_0} &= 2 \\ f_0(t) &= \frac{1}{\sqrt{2}}[rect(t) + rect(t-2)] \end{aligned}$$

Determining the second basis function:

$$\begin{aligned} c_{10} &= 0 \\ g_1(t) &= s_1(t) - c_{10} \times f_0(t) \\ &= rect(t-1) + rect(t-3) \\ \Rightarrow E_{g_1} &= 2 \end{aligned}$$

Therefore

$$\begin{aligned} f_1(t) &= \frac{g_1(t)}{\sqrt{E_{g_1}}} \\ &= \frac{1}{\sqrt{2}}[rect(t-1) + rect(t-3)] \end{aligned}$$

Determining the third basis function:

$$c_{20} = \frac{1}{\sqrt{2}}, c_{21} = \frac{1}{\sqrt{2}}$$

$$g_2(t) = \frac{1}{2}[-rect(t) + rect(t-1) + rect(t-2) - rect(t-3)]$$

$$E_{g_2} = 1$$

Therefore

$$f_2(t) = \frac{1}{2}[-rect(t) + rect(t-1) + rect(t-2) - rect(t-3)]$$

Determining the fourth basis function:

$$c_{30} = 0, c_{31} = 0, c_{32} = 1$$

$$g_3(t) = \frac{1}{2}[rect(t) + rect(t-1) - rect(t-2) - rect(t-3)]$$

$$E_{g_3} = 1$$

$$f_3(t) = \frac{1}{2}[rect(t) + rect(t-1) - rect(t-2) - rect(t-3)]$$

Using the basis functions $\{f_i(t)\}$, the signals can be represented in vector form as

$$\begin{aligned}
\underline{s}_0 &= [\sqrt{2}\ \ 0\ \ 0\ \ 0] \\
\underline{s}_1 &= [0\ \ \sqrt{2}\ \ 0\ \ 0] \\
\underline{s}_2 &= [\frac{1}{\sqrt{2}}\ \ \frac{1}{\sqrt{2}}\ \ 1\ \ 0] \\
\underline{s}_3 &= [0\ \ 0\ \ 1\ \ 1].
\end{aligned}$$

(b) Choosing orthonormal basis functions by inspection:

$$\begin{aligned}
b_0(t) &= rect(t) \\
b_1(t) &= rect(t-1) \\
b_2(t) &= rect(t-2) \\
b_3(t) &= rect(t-3)
\end{aligned}$$

Using the basis functions $\{b_i(t)\}$, the signals can be represented in vector form as

$$\begin{aligned}
\underline{s}_0 &= [1\ \ 0\ \ 1\ \ 0] \\
\underline{s}_1 &= [0\ \ 1\ \ 0\ \ 1] \\
\underline{s}_2 &= [0\ \ 1\ \ 1\ \ 0] \\
\underline{s}_3 &= [0\ \ 1\ \ 0\ \ -1].
\end{aligned}$$

(ii) In order to verify that one constellation can be obtained from the other by a rotation, we express the basis functions $\{f_i(t)\}$ as a linear combination of the basis functions $\{b_i(t)\}$. Then, we show that the linear transformation $\mathbf{C}$ is unitary, i.e., $\mathbf{C}^T\mathbf{C} = \mathbf{I}$.

$$\begin{aligned} f_0(t) &= \frac{1}{\sqrt{2}}[b_0(t) + b_2(t)] \\ f_1(t) &= \frac{1}{\sqrt{2}}[b_1(t) + b_3(t)] \\ f_2(t) &= \frac{1}{2}[-b_0(t) + b_1(t) + b_2(t) - b_3(t)] \\ f_3(t) &= \frac{1}{2}[b_0(t) + b_1(t) - b_2(t) - b_3(t)] \end{aligned}$$

or equivalently,

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} b_0(t) \\ b_1(t) \\ b_2(t) \\ b_3(t) \end{bmatrix} = \begin{bmatrix} f_0(t) \\ f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

$$\text{i.e. } \mathbf{C}\underline{\mathbf{b}} = \underline{\mathbf{f}}$$

where i^{th} element of $\underline{\mathbf{b}}$ is $b_i(t)$ and i^{th} element of $\underline{\mathbf{f}}$ is $f_i(t)$.

$$\mathbf{C}^{\mathbf{T}}\mathbf{C} = \mathbf{I}.$$

(iii) The distance of each point from the origin and the relative distances between the signal points remain the same in either case. (because of (ii)).

2. The minimum number of orthonormal basis functions required can be determined using Gram-Schmidt procedure. The answer is 2.

   Alternately, the minimum number of orthonormal basis functions can be determined as follows:

   - $s_0(t)$ and $s_1(t)$ are linearly independent. Therefore, at least 2 basis functions are required.
   - $s_2(t) = \frac{1}{4}s_0(t) - \frac{1}{2}s_1(t)$ and $s_3(t) = -s_0(t) - s_1(t)$, i.e., $s_2(t)$ and $s_3(t)$ can be expressed as linear combinations of $s_0(t)$ and $s_1(t)$. Therefore, no more than 2 basis functions are necessary.

3. (a) The waveforms $f_1(t)$, $f_2(t)$, and $f_3(t)$ are orthonormal because

$$\int_{-\infty}^{\infty} f_i(t) f_j(t) dt = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

(b) If $x(t)$ lies in the signal space spanned by $f_1(t)$, $f_2(t)$, and $f_3(t)$, it can be represented exactly as

$$x(t) = \sum_{i=1}^{3} x_i f_i(t),$$

where

$$x_i = \int_{-\infty}^{\infty} x(t) f_i(t) dt.$$

If $x(t)$ does not lie in the signal space spanned by $f_1(t)$, $f_2(t)$, and $f_3(t)$, it can be approximated as

$$\hat{x}(t) = \sum_{i=1}^{3} x_i f_i(t),$$

where

$$x_i = \int_{-\infty}^{\infty} x(t) f_i(t) dt$$

and

$$\int_{-\infty}^{\infty} [x(t) - \hat{x}(t)]^2 dt$$

is minimized.

In this case, we have $x(t) = 2f_1(t) + f_2(t) - 3f_3(t)$.

4. One signal constellation representation of the two signals $s_1(t)$ and $s_2(t)$ is shown in Figure 1. Gram-Schmidt procedure can be used to determine the basis functions.

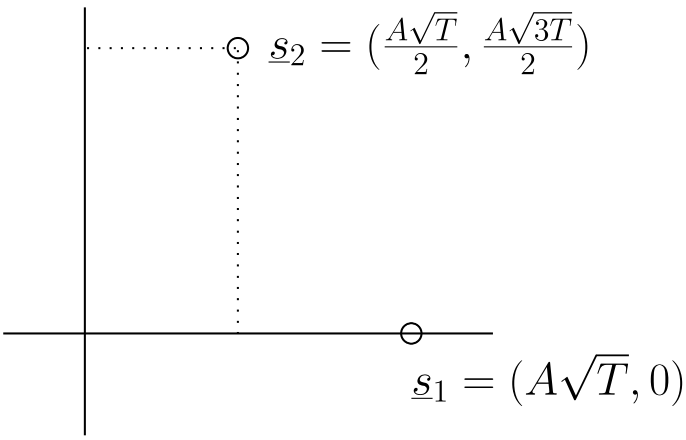

Figure 1: Signal Constellation

5. The decision regions for the optimal receiver are shown in Figure 2.

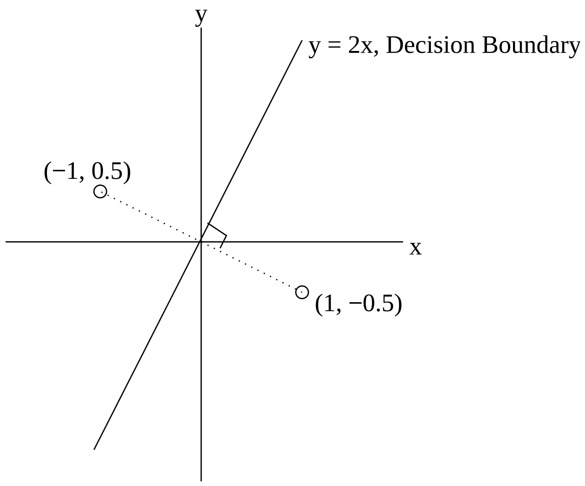

Figure 2: Decision Regions