# Jointly Optimal Power Control and Routing for a Single Cell, Dense, Ad Hoc Wireless Network*

Venkatesh Ramaiyan
Dept. of ECE
Indian Institute of Science
Bangalore, India
rvenkat@ece.iisc.ernet.in

Anurag Kumar
Dept. of ECE
Indian Institute of Science
Bangalore, India
anurag@ece.iisc.ernet.in

Eitan Altman
INRIA
Sophia-Antipolis
France
eitan.altman@sophia.inria.fr

*Abstract*—We consider a dense, ad hoc wireless network confined to a small region, such that direct communication is possible between any pair of nodes. The physical communication model is that a receiver decodes the signal from a single transmitter, while treating all other signals as interference. Data packets are sent between source-destination pairs by multihop relaying. We assume that nodes self-organise into a multihop network such that all hops are of length $d$ meters, where $d$ is a design parameter. There is a contention based multiaccess scheme, and it is assumed that every node always has data to send, either originated from it or a transit packet (saturation assumption). In this scenario, we seek to maximize a measure of the transport capacity of the network (measured in bit-meters per second) over power controls (in a fading environment) and over the hop distance $d$, subject to an average power constraint.

We first argue that for a dense collection of nodes confined to a small region, single cell operation is efficient for single user decoding transceivers. Then, operating the dense ad hoc network (described above) as a single cell, we study the optimal hop length and power control that maximizes the transport capacity for a given network power constraint. More specifically, for a fading channel and for a fixed transmission time strategy (akin to the IEEE 802.11 TXOP), we find that there exists an intrinsic aggregate bit rate ($\Theta_{opt}$ bits per second, depending on the contention mechanism and the channel fading characteristics) carried by the network, when operating at the optimal hop length and power control. The optimal transport capacity is of the form $d_{opt}(\bar{P}_t) \times \Theta_{opt}$ with $d_{opt}$ scaling as $\bar{P}_t^{\frac{1}{\eta}}$, where $\bar{P}_t$ is the available time average transmit power and $\eta$ is the path loss exponent. Under certain conditions on the fading distribution, we then provide a simple characterisation of the optimal operating point.

*Index Terms*—Multihop Relaying, Optimal Power Control, Self-Organisation, Fixed Transmission Time

## I. INTRODUCTION

We consider a scenario in which there is a large number of stationary nodes (e.g., hundreds of nodes) confined to a small area (e.g., spatial diameter 30m), and organised in to a multihop ad hoc wireless network. Source-destination pairs are chosen randomly and we assume that the traffic in the network is homogeneous. We assume that data packets are sent between source-destination pairs by multihop relaying with single user decoding and forwarding of packets, i.e., we assume that signals received from nodes other than the intended transmitter are treated as interference. A distributed multiaccess contention

scheme is used in order to schedule transmissions; for example, the CSMA/CA based distributed coordination function (DCF) of the IEEE 802.11 standard for wireless local area networks (WLANs). We assume that all nodes can decode all the contention control transmissions (i.e., there are no hidden nodes), and only one successful transmission takes place at any time in the network. In this sense we say that we are dealing with a *single cell* scenario. Thus our work in this paper can be viewed as an extension of the performance analysis presented in [2] and extended in [1]. We further assume that, during the exchange of contention control packets, pairs of communicating nodes are able to estimate the channel fading between themselves and are thus able to perform power control per transmission.

There is a natural tradeoff between using high power and long hop lengths (single hop direct transmission between the source-destination pair), versus using low power and shorter hop lengths (multihop communication using intermediate nodes), with the latter necessitating more packets to be transported in the network. The objective of the present paper is to study optimal routing, in terms of the hop length, and optimal power control for a fading channel, when a single cell network (such as that studied in [1]) is used in a multihop mode. Our objective is to maximise a certain measure of network transport capacity (measured in bit-meters per second; see Section IV), subject to a network power constraint. A network power constraint determines, to a first order, the lifetime of the network.

Situations and considerations such as those that we study could arise in a dense ad hoc sensor network. Ad hoc sensor networks are now being studied as possible replacements for wired measurement networks in large factories. For example, a distillation column in a chemical plant could be equipped with pressure and temperature sensors and valve actuators. The sensors monitor the system and communicate the pressure and temperature values to a central controller which in turn actuates the valves to operate the column at the desired operating point. Direct communication between the sensors and actuators is also a possibility. Such installations could involve hundreds of devices, organised into a single cell ad hoc wireless network because of the physical proximity of the nodes. There would be many flows within the network and there would be multihopping. We wish to address the question of optimal organisation of such an ad hoc network so as to maximise its transport capacity subject to a power constraint. The power constraint relates

to the network life-time and would depend on the application. In a factory situation, it is possible that power could be supplied to the devices, hence large power would be available. In certain emergencies, "transient" sensor networks could be deployed for situation management; we use the term "transient" as these networks are supposed to exist for only several minutes or hours, and the devices could be disposable. Such networks need to have large throughputs, but, being transient networks, the power constraint could again be loose. On the other hand sensor networks deployed for monitoring some phenomenon in a remote area would have to work with very small amounts of power, while sacrificing transport capacity. Our formulation aims at providing insights into optimal network operation in a range of such scenarios.

### A. Preview of Contributions

We motivate the definition of the transport capacity of the network as the product of the aggregate throughput (in bits per second) and the hop distance (in meters). For random spatio-temporal fading, we seek the power control and the hop distance that jointly optimise the transport capacity, subject to a network average power constraint. For a fixed data transmission time strategy (discussed in Section III-B), we show that the optimal power allocation function has a water pouring form (Section V-A). At the optimal operating point (power control and hop distance) the network throughput ($\Theta_{opt}$, in bits per second) is shown to be a fixed quantity, depending only on the contention mechanism and fading model, but independent of the network power constraint (Section V-B). Further, we show that the optimal transport capacity is of the form $d_{opt}(\bar{P}_t) \times \Theta_{opt}$, with $d_{opt}$ scaling as $\bar{P}_t^{\frac{1}{\eta}}$, where $\bar{P}_t$ is the available time average transmission power, and $\eta$ is the power law path loss exponent (Theorem V.2). Finally, we provide a condition on the fading density that leads to a simple characterisation of the optimal hop distance (Section V-C).

## II. Motivation for Single Cell Operation

In this context, the seminal paper by Gupta and Kumar [4] would suggest that each node should communicate with neighbours as close as possible while maintaining network connectivity. This maximises network transport capacity (in bit-metres per second), while minimising network average power. It has been observed by Dousse and Thiran [5], however, that if, unlike [4], the practical model of bounded received power for finite transmitter power is used, then the increasing interference with an increasing density of simultaneous transmitters is not consistent with a minimum SINR requirement at each receiver. The following discussion motivates a single cell operation in our framework, i.e., only one transmission exists in the network at any time.

Consider a dense planar wireless network with $n$ nodes in a square of fixed area $A$. Let $K(n)$ denote the spatial reuse in the network (number of simultaneous transmissions) with $r(K(n))$ bounding the transmitter-receiver separation. Let $P(K(n))$ be the transmitter power per node (with a network average power $\bar{P}$, as in Section V) and let $N$ be the receiver noise power. The maximum SINR achievable per link in such a network

(with single user decoding receivers) is bounded, i.e., $SINR \leq \frac{P(K(n))}{(N+I_{K(n)})}$, where $I_{K(n)}$ denotes the interference at a node due to spatial reuse. Using the finite (and fixed) area assumption, we bound the minimum interference by any simultaneous transmission by $\frac{P(K(n))}{(\sqrt{2A})^\eta}$. Hence, $SINR \leq \frac{P(K(n))}{N+(K(n)-1)\frac{P(K(n))}{(2A)^{\frac{\eta}{2}}}}$. The aggregate capacity (bits/slot) achieved in such a network is now bounded above by

$$C(K(n)) := K(n) \log \left( 1 + \frac{P(K(n))}{N + (K(n)-1)\frac{P(K(n))}{(2A)^{\frac{\eta}{2}}}} \right)$$

Clearly, $C(K(n))$ is uniformly bounded for any $K(n)$. Since $r(K(n)) \leq \sqrt{2A}$, we see that the transport capacity (in bit-metres per second, see Section IV) achieved in the network, bounded above by $C(K(n))r(K(n))$, is also finite, independent of the number of nodes and power $\bar{P}$. Also, we expect that the transmitter-receiver separation (bounded above by $r(K(n))$) would decrease to 0 as $K(n)$ increases to $\infty$ (finite area assumption). Hence, $\lim_{K(n)\to\infty} C(K(n))r(K(n)) = 0$. This implies that there exists an optimal $K(n), 1 < K(n) < \infty$, which maximises the transport capacity in the network, i.e., the optimum spatial reuse is finite. However, we note that, without spatial reuse, a simple TDMA scheme with direct transmissions between the source and the destination with transmit power $n\bar{P}$ (and hence, an average power $\bar{P}$), achieves $\log(n)$ order transport capacity. As seen above, with spatial reuse, the system becomes interference limited, and hence, becomes inefficient both for large $n$ and for large $\bar{P}$. Thus, we conclude that single cell operation (as defined earlier) is efficient for dense networks with single user decoding. In the context of sensor networks, $\log(n)$ scaling has been achieved with maximum node power constraints as well, using cooperative transmission techniques ([6]).

With the above motivation, in this work, we study the transport capacity of power constrained dense ad hoc networks operated as a single cell. More recently, El Gamal and Mammen [7] have shown that, if the transceiver energy at each hop is factored in, then the operating regime studied in [4] is neither energy efficient nor delay optimal. Fewer hops between the transmitter and receiver (and hence, less spatial reuse) reduce the energy consumption and lead to a better throughput-delay tradeoff. While optimal operation of the network might suggest using some spatial reuse (finite, as discussed above), coordinating simultaneous transmissions (in a distributed fashion), in a constrained area, is extremely difficult and the associated time, energy and synchronisation overheads have to be accounted for. In view of the above discussions, in this paper, we assume that the multiple access control (MAC) is such that only one transmitter-receiver pair communicate at any time in the network.

### A. Outline of the Paper

In Section III we describe the system model and in Section IV we motivate the objective. We study the transport capacity of a single cell multihop wireless network, operating in the fixed transmission time mode, in Section V. Section VI concludes

the paper and discusses future work. *The proofs of all lemmas and theorems, if not in the paper, are provided in [10].*

## III. THE NETWORK MODEL

There is a dense collection of immobile nodes that use multiaccess multihop radio communication with single user decoding and packet forwarding to transport packets between various source-destination pairs.

- All nodes use the same contention mechanism with the same parameters (e.g., all nodes use IEEE 802.11 DCF with the same back-off parameters).
- We assume that nodes send control packets (such as RTS/CTS in IEEE 802.11) with a constant power (i.e., power control is not used for the control packets) during contention, and these control packets are decodable by *every* node in the network. As in IEEE 802.11, this can be done by using a low rate, robust modulation scheme and by restricting the diameter of the network. This is the "single cell" assumption, also used in [1], and implies that there can be only one successful ongoing transmission at any time.
- During the control packet exchange, each transmitter learns about the channel "gain" to its intended receiver, and decides upon the power level that is used to transmit its data packet. For example, in IEEE 802.11, the channel gain to the intended receiver could be estimated during the RTS/CTS control packet exchange. Such channel information can then be used by the transmitter to do power control. In our paper, we assume that such channel estimation and power control is possible on a transmission-by-transmission basis.
- In this work, we model only an average power constraint and not a peak power constraint.
- We assume that the traffic is homogeneous in the network and all the nodes have data to send at all times; these could be locally generated packets or transit packets (saturation assumption).

### A. Channel Model: Path Loss, Fading and Transmission Rate

The channel gain between a transmitter-receiver pair for a hop is a function of the hop length and the multipath fading "gain" ($h$). Based on our dense network and traffic homogeneity assumption, we further make the following assumption.

- The nodes self-organise so that all hops are of length $d$, i.e., a one hop transmission always traverses a distance of $d$ meters. This hop distance, $d$, will be one of our optimisation variables.

The path loss for a hop distance $d$ is given by $\frac{1}{d^\eta}$, where $\eta$ is the path loss exponent, chosen depending on the propagation characteristics of the environment (see, for e.g., [8]). This variation of path loss with $d$ holds for $d > d_0$, the far field reference distance; we will assume that this inequality holds ($d > d_0$), and will justify this assumption in the course of the analysis below (see Theorem V.2).

We assume that for each transmitter-receiver pair, the channel gain due to multipath fading may change from transmission to transmission, but remains constant over any packet transmission duration. Since successive transmissions can take place between randomly selected pairs of nodes (as per the outcome of the distributed contention mechanism) we are actually modeling a spatio-temporal fading process. We assume that this fading process is stationary in space and time with some given marginal distribution $H$. Let the cumulative distribution of $H$ be $A(h)$ (with a p.d.f. $a(h)$), which by our assumption of spatio-temporal stationarity of fading is the same for all transmitter-receiver pairs and for all transmissions. We assume a flat and slow fading channel with additive white Gaussian noise of power $\sigma^2$. And, $\tau_c$, the channel coherence time applicable to all the links in the network, upper bounds the time taken to complete any data transmission in the network. We assume that $H$ and $\tau_c$ are independent of the hop distance $d$.

When a node transmits to another node at a distance $d$ (in the transmitting antenna's far field), using transmitter power $P$, with channel power gain due to fading, $h$, then we assume that the transmission rate given by Shannon's formula is achieved over the transmission burst; i.e., the transmission rate is given by

$$C = W \log\left(1 + \frac{hP\alpha}{\sigma^2 d^\eta}\right)$$

where $W$ is the signal bandwidth and $\alpha$ is a constant accounting for any fixed power gains between the transmitter and the receiver. Note that this requires that the transmitter has available several coding schemes of different rates, one of which is chosen for each channel state and power level.

### B. Fixed Transmission Time Strategy

We focus on a fixed transmission time scheme, where all data transmissions are of equal duration, independent of the bit rate achieved over the wireless link. This implies that the amount of data that a transmitter sends during a transmission opportunity is proportional to the achieved physical link rate. Let $T$ ($< \tau_c$, the channel coherence time), be the data transmission time. Upon a successful control packet exchange, the channel (between the transmitter, that "won" the contention, and its intended receiver) is reserved for a duration of $T$ seconds independent of the channel state $h$. This is akin to the "TxOP" (transmission opportunity) mechanism in the IEEE 802.11 standard. Thus, when the power allocated during the channel state $h$ is $P(h)$, and $P(h) > 0$, then data transmission occupies the channel for the duration $T$ seconds, sending $C(h)T$ bits across the channel, where $C(h) = W \log\left(1 + \frac{P(h)h\alpha}{\sigma^2 d^\eta}\right)$. If $P(h) = 0$, we assume that the channel is left idle for the next $T$ seconds.

The optimality of a fixed transmission time scheme, for throughput, as compared to a fixed packet length scheme, can be formally established (see [10]), but, due to lack of space, we only provide an intuition here. When using fixed packet lengths, a transmitter may be forced to send the entire packet even if the channel is poor, thus taking longer time and more power. On the other hand, in a fixed transmission time scheme, we send more data when the channel is good and limit our inefficiency when the channel is poor.

## IV. Multihop Transport Capacity

Let $d$ denote the common hop length and $\{P(h)\}$ a power allocation policy, with $P(h)$ denoting the transmit power used when the channel state is $h$. We take a simple model for the random access channel contention process. The channel goes through successive contention periods. Each period can be either an idle slot, or a collision period, or a successful transmission with probabilities $p_i, p_c$ and $p_s$ respectively. Under the node saturation assumption, the aggregate bit rate carried by the system, $\Theta_T(\{P(h)\}, d)$, for the hop distance $d$ and power allocation $\{P(h)\}$, is given by (see [2], or [1])

$$\Theta_T(\{P(h)\}, d) := \frac{p_s(\int_0^\infty L(h) \, dA(h) \,)}{p_i T_i + p_c T_c + p_s(T_o + T)} \qquad (1)$$

where $L(h) = C(h)T$, and, $T_i, T_c$ and $T_o$ are the average time overheads associated with an idle slot, collision and data transmission. For e.g., in IEEE 802.11 with the RTS/CTS mechanism being used, a collision takes a fixed time independent of the data transmission rate. We note that $p_i, p_s, p_c, T_i, T_o$, and $T_c$ depend only on the parameters of the distributed contention mechanism (MAC protocol), and not on any of the decision variables that we consider.

With $\Theta_T(\{P(h)\}, d)$ defined as in (1), we consider $\Theta_T(\{P(h)\}, d) \times d$ as our measure of transport capacity of the network. This measure can be motivated in several ways. $\Theta_T(\{P(h)\}, d)$ is the rate at which bits are transmitted by the network nodes. When transmitted successfully, each bit traverses a distance $d$. Hence, $\Theta_T(\{P(h)\}, d) \times d$ is the rate of spatial progress of the flow of bits in the network (in bit-metres per second). Viewed alternatively, it is the weighted average of the end-to-end flow throughput with respect to the distance traversed. Suppose that a flow $i$ covers a distance $D_i$ with $\frac{D_i}{d}$ hops (assumed to be an integer for this argument). Let $\beta_i \Theta_T(\{P(h)\}, d)$ be the fraction of throughput of the network that belongs to flow $i$. Then, $\frac{\beta_i \Theta_T(\{P(h)\}, d)}{\frac{D_i}{d}}$ is the end-to-end throughput for flow $i$ and $\frac{\beta_i \Theta_T(\{P(h)\}, d)}{\frac{D_i}{d}} \times D_i = \beta_i \Theta_T(\{P(h)\}, d) \times d$ is the end-to-end flow throughput for flow $i$ in bit-metres per second. Summing over all the flows, we have $\Theta_T(\{P(h)\}, d) \times d$, the aggregate end-to-flow throughput in bit-metres per second.

With the above motivation, our aim in this paper is to maximise the quantity $\Theta_T(\{P(h)\}, d) \times d$ over the hop distance $d$ and over the power control $\{P(h)\}$, subject to a *network average power constraint*, $\bar{P}$. We use a network power constraint that accounts for the energy used in data transmission as well as the energy overheads associated with communication.

## V. Optimising the Transport Capacity

For a given $\{P(h)\}$ and $d$, and the corresponding throughput $\Theta_T(\{P(h)\}, d)$, the transport capacity in bit-meters per second, which we will denote by $\psi(\{P(h)\}, d)$, is given by

$$\psi(\{P(h)\}, d) := \Theta_T(\{P(h)\}, d) \times d$$

Maximizing $\psi(\cdot, \cdot)$ involves optimizing over $d$, as well as $\{P(h)\}$. However, we observe that, it would not be possible

to vary $d$ with fading, as routes cannot vary at the fading time scale. Hence, we propose to optimize first over $\{P(h)\}$ for a given $d$, and then optimize over $d$, i.e., we seek to solve the following problem,

$$\max_d \max_{\{\{P(h)\}: \mathcal{P}(\{P(h)\}) \leq \bar{P}\}} \psi(\{P(h)\}, d) \qquad (2)$$

where the network average power, $\mathcal{P}(\{P(h)\})$, is given by,

$$\mathcal{P}(\{P(h)\}) := $$
$$\frac{p_i E_i + p_c E_c + p_s(E_o + T \int_0^\infty P(h) \, dA(h) \,)}{p_i T_i + p_c T_c + p_s(T_o + T)} \qquad (3)$$

$E_i, E_c$ and $E_o$ correspond to the energy overheads associated with an idle period, collision and successful transmission. Thus, $E_i$ denotes the total energy expended in the network over an idle slot, $E_c$ denotes the total average energy expended by the colliding nodes, as well as the idle energy of the idle nodes, and $E_o$ denotes the average energy expended in the successful contention negotiation between the successful transmitter-receiver pair, the receive energy at the receiver (in the radio and in the packet processor), and the idle energy expended by all the other nodes over the time $T_o + T$.

For a given $d$ and power allocation $\{P(h)\}$, define the time average transmission power, $\bar{P}_t(\{P(h)\}, d)$, and the time average overhead power, $\bar{P}_o$ (which does not depend on $\{P(h)\}$ or $d$), as

$$\bar{P}_t(\{P(h)\}, d) := \frac{p_s(\int_0^\infty P(h) \, dA(h) \,)T}{p_i T_i + p_c T_c + p_s(T_o + T)}$$

$$\bar{P}_o := \frac{p_i E_i + p_c E_c + p_s E_o}{p_i T_i + p_c T_c + p_s(T_o + T)}$$

Then the network power constraint can be rewritten as

$$\bar{P}_t(\{P(h)\}, d) \leq \bar{P} - \bar{P}_o$$

where the right hand side does not depend on $\{P(h)\}$ or $d$. Observe that $\bar{P}_t(:= \bar{P} - \bar{P}_o)$ is the *time average* transmission power constraint.

### A. Optimization over $\{P(h)\}$ for a fixed $d$

Consider the optimization problem

$$\max_{\{\{P(h)\}: \mathcal{P}(\{P(h)\}) \leq \bar{P}\}} \psi(\{P(h)\}, d) \qquad (4)$$

The denominators of $\Theta_T(\cdot, \cdot)$ in (1) and of $\mathcal{P}$ in (3) are independent of $d$ and the power control $\{P(h)\}$. Thus, with $d$ fixed, the optimization problem simplifies to maximizing $\int_0^\infty L(h) \, dA(h)$ or,

$$\int_0^\infty \log\left(1 + \frac{P(h)h\alpha}{\sigma^2 d^\eta}\right) \, dA(h)$$

subject to the average power contraint,

$$\int_0^\infty P(h) \, dA(h) \leq \bar{P}_t{}'$$

where $\bar{P}_t{}'$ is given by,

$$\bar{P}_t{}' := \frac{(p_i T_i + p_c T_c + p_s(T_o + T))}{p_s T} \bar{P}_t$$

Notice that $\bar{P}_t{}'$ is also independent of $\{P(h)\}$ or $d$ and is the average transmit power constraint averaged only over the transmission periods.

Without a peak power constraint, this is a well-known problem whose optimal solution has the water-pouring form (see [3]). The optimal power allocation function $\{P(h)\}$ is given by

$$P(h) = \left(\frac{1}{\lambda} - \frac{d^\eta \sigma^2}{h\alpha}\right)^+$$

where $\lambda$ is obtained from the power constraint equation

$$\int_{\frac{\lambda \sigma^2 d^\eta}{\alpha}}^{\infty} a(h) P(h) dh = \bar{P}_t{}'$$

The optimal power allocation is a nonrandomized policy, where a node transmits with power $P(h)$ every time the channel is in state $h$ (whenever $P(h) > 0$), or leaves the channel idle for $h$ such that $P(h) = 0$.

*B. Optimization over $d$*

By defining $\Phi(h) := \frac{P(h)}{d^\eta}$, the problem of maximising the throughput over power controls, for a fixed $d$, becomes

$$\max \int_0^\infty \log\left(1 + \frac{\alpha h}{\sigma^2}\Phi(h)\right) a(h) dh$$

subject to

$$\int_0^\infty \Phi(h) a(h) dh \leq \frac{\bar{P}_t{}'}{d^\eta}$$

Denoting by $\Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right)$ the optimal value of this problem, the problem of optimisation over the hop-length now becomes

$$\max_d d \times \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right) \quad (5)$$

*Theorem V.1:* In the problem defined by (5), the objective $d \times \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right)$, when viewed as a function of $d$, is continuously differentiable. Further, when the channel fading random variable, $H$, has a finite mean ($E(H) < \infty$), then

1) $\lim_{d \to 0} d \times \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right) = 0$ and,

2) if in addition, $\eta \geq 2$, $\frac{1}{h^2} a\left(\frac{1}{h}\right)$ is continuously differentiable and $P(H > h) = O\left(\frac{1}{h^2}\right)$ for large $h$, then, $\lim_{d \to \infty} d \times \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right) = 0$,

*Proof:* The proofs of continuous differentiability of $d \times \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right)$, 1) and 2) are provided in [10]. ∎

*Remarks V.1:*

1) Under the conditions proposed in Theorem V.1, it follows that $d \times \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right)$ is bounded over $d \in [0, \infty)$ and achieves its maximum in $d \in (0, \infty)$.

2) When the objective function (5) is unbounded, the optimal solution occurs at $d = \infty$ (from the continuity results).

3) We note that, in practice, $\eta \geq 2$.

*Theorem V.2:* The following hold for the problem in (5),

1) Without the constraint $d > d_0$, the optimum hop distance $d_{opt}$ scales as $(\bar{P}_t{}')^{\frac{1}{\eta}}$.

2) There is a value $\bar{P}_t{}'_{\min}$ such that, for $\bar{P}_t{}' > \bar{P}_t{}'_{\min}$, $d_{opt} > d_0$, and hence the optimal solution obeys the scaling shown in 1).

3) For $\bar{P}_t{}' > \bar{P}_t{}'_{\min}$, the optimum power control $\{P(h)\}$ is of the water pouring form and scales as $\bar{P}_t{}'$.

4) For $\bar{P}_t{}' > \bar{P}_t{}'_{\min}$, the optimal transport capacity scales as $(\bar{P}_t{}')^{\frac{1}{\eta}}$.

*Proof:*

1) Let $d_{opt}$ be optimal for $\bar{P}_t{}' > 0$. We claim that, for $x > 0$, $x^{\frac{1}{\eta}} d_{opt}$ is optimal for the power constraint $x\bar{P}_t{}'$. For suppose this was not so, it would mean that there exists $d > 0$ such that

$$\left(x^{\frac{1}{\eta}} d_{opt} \Gamma\left(\frac{x\bar{P}_t{}'}{(x^{\frac{1}{\eta}} d_{opt})^\eta}\right)\right) < d \Gamma\left(\frac{x\bar{P}_t{}'}{d^\eta}\right)$$

or, equivalently,

$$\left(d_{opt} \Gamma\left(\frac{\bar{P}_t{}'}{d_{opt}^\eta}\right)\right) < x^{-\frac{1}{\eta}} d \Gamma\left(\frac{\bar{P}_t{}'}{(x^{-\frac{1}{\eta}} d)^\eta}\right)$$

which contradicts the hypothesis that $d_{opt}$ is optimal for $\bar{P}_t{}'$.

2) Using the path loss model $\frac{P}{d^\eta}$, we see that for $d < d_0$, the received power is scaled more than $P$, due to the factor $\frac{P}{d^\eta}$, and an $d_0^\eta$ factor in $\alpha$, i.e., the model over-estimates the received power and the transport capacity. Hence, the achieved transport capacity for $d < d_0$ is definitely less than $d \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right)$. The result now follows from the scaling result in 1).

3) It follows from 1) that, if $\bar{P}_t{}'$ scales by a factor $x$, then the optimum $d$ scales by $x^{\frac{1}{\eta}}$, so that, at the optimum, $\frac{\bar{P}_t{}'}{d^\eta}$ is unchanged. Hence the optimal $\{\Phi(h)\}$ is unchanged, which means that $\{P(h)\}$ must scale by $x$. The water pouring form is evident.

4) Again, by 1) and 2), if $\bar{P}_t{}'$ scales by a factor $x$, then the optimum $d$ scales by $x^{\frac{1}{\eta}}$, so that, at the optimum, $\frac{\bar{P}_t{}'}{d^\eta}$ is unchanged. Thus $\Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right)$ is unchanged, and the optimal transport capacity scales as the optimum $d$, i.e., by the factor $x^{\frac{1}{\eta}}$. ∎

*Remarks V.2:*

The above theorem yields the following observations for the fixed transmission time model.

1) As an illustration, with $\eta = 3$, in order to double the optimal transport capacity, we need to use $2^3$ times the $\bar{P}_t{}'$. This would result in a considerable reduction in network lifetime, assuming the same battery energy.
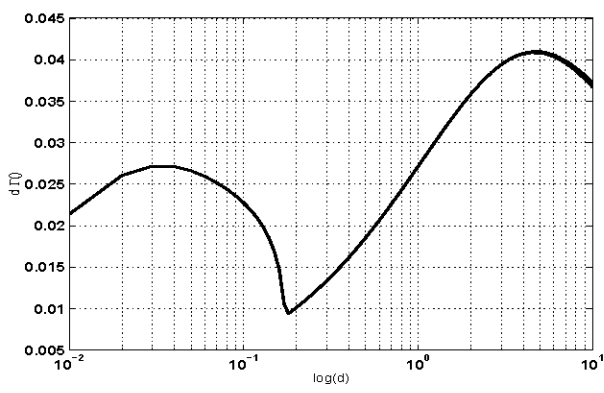
Fig. 1. Plot of $d \times \Gamma\left(\frac{1}{d^3}\right)$ (linear scale) vs. $d$ (log scale) for a channel with two fading states $h_1, h_2$. The fading gains are $h_1 = 100$ and $h_2 = 0.5$, with probabilities $a_{h_1} = 0.01 = 1 - a_{h_2}$. The function has 3 non-trivial stationary points.

2) We observe that as the power constraint $\bar{P}_t{}'$ scales, the optimal bit rate carried in the network, $\Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right)$, stays constant, but the optimal transport capacity increases since the optimal hop length increases. Further, because of the way the optimal power control and the optimal hop length scale together, the nodes transmit at the same *physical bit rate* in each fading state; see the proof of Theorem V.2 part 3). ∎

### C. Characterisation of the Optimal d

By the results in Theorem V.1 we can conclude that the optimal solution of the maximisation in (5) lies in the set of points for which the derivative of $d \times \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right)$ is zero. For fixed $\bar{P}_t{}'$, define $\pi(d) := \frac{\bar{P}_t{}'}{d^\eta}$. Differentiating $d \times \Gamma(\pi(d))$, we obtain, (see Appendix A)

$$\frac{\partial}{\partial d}(d\,\Gamma(\pi(d)) = \Gamma(\pi(d)) - \eta\pi(d)\lambda(\pi(d))$$

where $\lambda(\pi)$ is the Lagrange multiplier for the optimisation problem that yields $\Gamma(\pi(d))$. Since $d$ appears only via $\pi(d)$, we can view the right hand side as a function of $\pi$. We are interested in the zeros of the above expression. Clearly, $\pi = 0$ is a solution. This solution corresponds to the case $d = \infty$; However, we are interested only in solutions of $d \in (0, \infty)$, and hence, we seek positive solutions of $\pi$ of $\Gamma(\pi) - \eta\pi\lambda(\pi) = 0$.

*Remarks V.3:* The above analysis has been done for a continuously distributed fading random variable $H$. The analysis can be done for a discrete valued fading distribution as well, and we provide this analysis in [10]. The following example then illustrates that, in general, the function $\Gamma(\pi) - \eta\pi\lambda(\pi) = 0$ can have multiple solutions. Consider a fading distribution that takes two values: $h_1 = 100$ and $h_2 = 0.5$, with probabilities $a_{h_1} = 0.01 = 1 - a_{h_2}$. Figure 1 plots $d \times \Gamma\left(\frac{1}{d^3}\right)$ for the system with $\eta = 3$. Notice that there are 3 stationary points other than the trivial solution $d = \infty$ (which is not shown in the figure). Also, the maximising solution is not the first stationary point (the stationary point close to 0). If, on the other hand, $a_{h_1} = 0.001 = 1 - a_{h_2}$, we again have 3 stationary points, but the optimal solution now is the first stationary point. ∎

More generally, and still pursuing the discrete case, let $\mathcal{H}$ denote the set of fading states when the fading random variable is discrete with a finite number of values; $|\mathcal{H}|$ denotes the cardinality of $\mathcal{H}$.

*Theorem V.3:* There are at most $2|\mathcal{H}| - 1$ stationary points of $d\,\Gamma(\pi(d))$ in $0 < d < \infty$.

*Proof:* See [10] for the related analysis and the proof of this theorem. ∎

We conclude from the above discussion that it is difficult to characterise the optimal solution when there are multiple stationary points. Hence we seek conditions for a unique positive stationary point, which must then be the maximising solution. In Appendix A, we have shown that the equation characterising the stationary points, $\Gamma(\pi) - \eta\pi\lambda(\pi) = 0$, can be rewritten as

$$\int_0^1 (\log(y) - \eta(y-1))\frac{\lambda^2}{y^2}f\left(\frac{\lambda}{y}\right)\,dy = 0 \qquad (6)$$

for $f(x) := a\left(\frac{\sigma^2 x}{\alpha}\right)\frac{\sigma^2}{\alpha}$, the density of the random variable $\frac{\alpha H}{\sigma^2}$. Notice that $\pi$ does not appear in this expression. The solution directly yields the Lagarange multiplier of the throughput maximisation problem for the optimal value of hop length. The following theorem guarantees the existence of atmost one stationary point of (6).

*Theorem V.4:* If for any $\lambda_1 > \lambda_2 > 0$, $\frac{f\left(\frac{\lambda_2}{y}\right)}{f\left(\frac{\lambda_1}{y}\right)}$ is a strictly monotonic decreasing function of $y$, then the objective function $d \times \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right)$ has at most one stationary point $d_{opt}, 0 < d_{opt} < \infty$.

*Proof:* The proof follows from Lemmas A.1, and A.2 in Appendix. ∎

*Corollary V.1:* If $H$ has an exponential distribution and $\eta \geq 2$, then the objective in the optimisation problem of (5) has a unique stationary point $d_{opt} \in (0, \infty)$, which achieves the maximum.

*Proof:* $a(h)$ is of the form $\mu e^{-\mu h}$ and the monotonicity hypothesis in Theorem V.4 holds for $a(h)$. Also, from Theorem V.1, we see that $\lim_{d \to 0} d \times \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right) = 0$ and $\lim_{d \to \infty} d \times \Gamma\left(\frac{\bar{P}_t{}'}{d^\eta}\right) = 0$. ∎

*Remarks V.4:* 1) Hence, for $\eta \geq 2$, for the Rayleigh fading model there exists a unique stationary point which corresponds to the optimal operating point.

2) For $\bar{P}_t{}' > \bar{P}_t{}'_{\min}$, and for the conditions in Theorem V.1 and V.4, let $\pi_{opt}$ denote the unique stationary point of (6). Then define $\Gamma(\pi_{opt}) = \Theta_{opt}$. It follows from Theorem V.2 that the optimal transport capacity takes the form $\left(\frac{\bar{P}_t{}'}{\pi_{opt}}\right)^{\frac{1}{\eta}}\Theta_{opt}$, where $\Theta_{opt}$ depends on $a(h)$ and the MAC parameters but not on $\bar{P}$ (or $\bar{P}_t$).

3) Figure 2 numerically illustrates our results for Rayleigh fading and $\eta = 2$. Scaling $\bar{P}_t{}'$ by 4 scales the transport capacity from 2.3 to 4.6, i.e., by $4^{\frac{1}{\eta}} = \sqrt{4}$ and similarly for scaling $\bar{P}_t{}'$ by 9.

The uniqueness results guarantees that a distributed implementation of the optimization problem, if it converges, shall converge to the unique stationary point, which is the optimal solution.
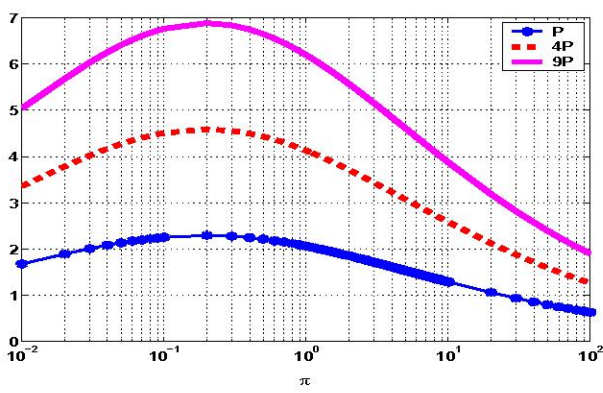
Fig. 2. Plot of $d \times \Gamma\left(\frac{\bar{P}_t{'}}{d^\eta}\right)$ (linear scale) vs. $\pi (= \frac{\bar{P}_t{'}}{d^\eta})$ (log scale) for a fading channel (with exponential distribution). We consider 3 power levels ($\bar{P}_t{'}$, $4\bar{P}_t{'}$ and $9\bar{P}_t{'}$) and $\eta = 2$. The function has a unique optimum $\pi_{opt}(\pi_{opt} \approx 0.2)$ for all the 3 cases.

## VI. Conclusion

In this paper we have studied a problem of jointly optimal power control and self-organisation in a single cell, dense, ad hoc multihop wireless network. The self-organisation is in terms of the hop distance used when relaying packets between source-destination pairs.

We formulated the problem as one of maximising the transport capacity of the network subject to an average power constraint. We showed that, for a fixed transmission time scheme, there corresponds an intrinsic aggregate packet carrying capacity at which the network operates at the optimal operating point, independent of the average power constraint. We also obtained the scaling law relating the optimal hop distance to the power constraint, and hence relating the optimal transport capacity to the power constraint (see Theorem V.2). Because of the way the power control and the optimal hop length scale, the optimal physical bit rate in each fading state is invariant with the power constraint. In Theorem V.4 we provide a characterisation of the optimal hop distance in cases in which the fading density satisfies a certain monotonicity condition.

One motivation for our work is the optimal operation of sensor networks. If a sensor network is supplied with external power, or if the network is not required to have a long life-time, then the value of the power constraint, $\bar{P}$, can be large, and a long hop distance will be used, yielding a large transport capacity. On the other hand, if the sensor network runs on batteries and needs to have a long life-time then $\bar{P}$ would be small, yielding a small hop length. In both cases the optimal aggregate bit rate carried by the network would be the same.

Future work on this topic will include developing a distributed algorithm for nodes to adapt themselves towards the optimal operating point.

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## Appendix

### A. Stationary Points of $d \times \Gamma(\pi(d))$

Recall that we defined $\pi(d) := \frac{\bar{P}_t{'}}{d^\eta}$. Further, $\Gamma(\pi(d))$ was defined by

$$\Gamma(\pi(d)) := \max \int_0^\infty \log\left(1 + \frac{\alpha h}{\sigma^2} \frac{P(h)}{d^\eta}\right) a(h) dh \quad (7)$$

where the maximum is over all power controls $\{P(h)\}$ satisfying the constraint

$$\int_0^\infty \frac{P(h)}{d^\eta} a(h) dh \leq \pi(d) \quad (8)$$

For ease of notation, let us use the substitution $x := \frac{\alpha h}{\sigma^2}$. Write $\xi(x) := \xi(\frac{\alpha h}{\sigma^2}) = \frac{P(h)}{d^\eta}$ and $f(x) := a\left(\frac{\sigma^2 x}{\alpha}\right)\frac{\sigma^2}{\alpha}$. Note that $f(\cdot)$ is the probability density of the random variable $X := \frac{\alpha H}{\sigma^2}$. Then, equations (7) and (8) can be rewritten as

$$\Gamma(\pi) = \max \int_0^\infty \log(1 + x\xi(x)) f(x) dx$$

and

$$\int_0^\infty \xi(x) f(x) dx \leq \pi$$

This optimisation problem is one of maximising a convex functional of $\{\xi(x)\}$, subject to a linear constraint. The optimal solution of the problem has water-pouring form, and the optimal solution is given by,

$$\xi(x) = \left(\frac{1}{\lambda(\pi)} - \frac{1}{x}\right)^+$$

where $\lambda(\pi)$ is obtained from

$$\int_{\lambda(\pi)}^\infty \left(\frac{1}{\lambda(\pi)} - \frac{1}{x}\right) f(x) dx = \pi$$

Further, the derivative of the optimum value $\Gamma(\pi)$, w.r.t. $\pi$, i.e., $\frac{\partial \Gamma(\pi)}{\partial \pi} = \lambda(\pi)$ (see Aubin [9]).

Let us now reintroduce the dependence on $d$, and consider the problem of optimising $d \times \Gamma(\pi(d))$ over $d$. Differentiating $d\, \Gamma(\pi(d))$ w.r.t. $d$, we get,

$$\begin{aligned}
\frac{\partial}{\partial d}(d\, \Gamma(\pi(d)) &= \Gamma(\pi(d)) + d\frac{\partial}{\partial d}\Gamma(\pi(d)) \\
&= \Gamma(\pi(d)) + d\frac{\partial \Gamma}{\partial \pi}(\pi(d)) \times \frac{\partial \pi(d)}{\partial d} \\
&= \Gamma(\pi(d)) + d\,\Gamma'(\pi(d)) \times \frac{-\eta \bar{P}_t{'}}{d^{\eta+1}} \\
&= \Gamma(\pi(d)) - \eta\pi(d)\Gamma'(\pi(d))
\end{aligned}$$

where $\Gamma'(\pi) := \frac{\partial \Gamma(\pi)}{\partial \pi}$. Substituting $\Gamma'(\pi) = \lambda(\pi)$, we have,

$$\frac{\partial}{\partial d}(d\Gamma(\pi(d)) = \Gamma(\pi(d)) - \eta\pi(d)\lambda(\pi(d)) \qquad (9)$$

The stationary points of $d \times \Gamma(\pi(d))$ are now obtained by equating the right hand side of (9) to zero. Note that since $d$ appears in this equation only as $\pi(d)$, we need only study the roots of the equation

$$\Gamma(\pi) - \eta\pi\lambda(\pi) = 0 \qquad (10)$$

We now proceed to obtain a characterisation of the stationary points. Substituting the optimal solution in the expression of $\Gamma(\pi)$ and $\lambda(\pi)$, and suppressing the argument $\pi$ in $\lambda(\pi)$, we get,

$$\Gamma(\pi) = \int_\lambda^\infty \log\left(\frac{x}{\lambda}\right) f(x)dx \qquad (11)$$

with $\lambda$ being given by

$$\pi = \int_\lambda^\infty \left(\frac{1}{\lambda} - \frac{1}{x}\right) f(x)dx \qquad (12)$$

Using the substitution $z = \frac{1}{x}$, $l = \frac{1}{\lambda}$, and defining $g(z) = \frac{1}{z^2}f\left(\frac{1}{z}\right)$, we get,

$$\Gamma(\pi) = \int_0^l \log\left(\frac{l}{z}\right) g(z)dz \qquad (13)$$

with $l$ (actually, $l(\pi)$) being given by

$$\pi = \int_0^l (l-z)\, g(z)dz \qquad (14)$$

We note that $g(\cdot)$ is the density of the random variable $Z := \frac{1}{X} = \frac{\sigma^2}{\alpha H}$.

For a function $t(\cdot)$ of the random variable $Z$, define the operators $\mathsf{E}_l(\cdot)$ and $\mathsf{G}_l(\cdot)$ as

$$\mathsf{E}_l(t(Z)) \quad := \quad \frac{\int_0^l t(z)g(z)dz}{\int_0^l g(z)dz}$$

$$\mathsf{G}_l(t(Z)) \quad := \quad \int_0^l t(z)g(z)dz$$

*Lemma A.1:* The roots of (10) are equivalent to obtaining the roots of the equation

$$\eta\mathsf{G}_{\frac{1}{\lambda}}(\lambda Z - 1) = \mathsf{G}_{\frac{1}{\lambda}}\left(\log(\lambda Z)\right) \qquad (15)$$

with $\pi$ then being given by

$$\pi = \int_\lambda^\infty \left(\frac{1}{\lambda} - \frac{1}{x}\right) f(x)dx$$

*Proof:* Using the definitions of $\mathsf{E}_l(\cdot)$ and $\mathsf{G}_l(\cdot)$, (13) and (14) simplify to

$$\Gamma(\pi) = \log(l)\mathsf{P}(Z \le l) - \mathsf{G}_l(\log(Z)) \qquad (16)$$

$$\pi = l\mathsf{P}(Z \le l) - \mathsf{G}_l(Z) \qquad (17)$$

(17) provides the $l$ (actually $l(\pi)$) to be substituted in (16). Substituting for $\Gamma(\pi)$ (from (16)), and for $l$ (from (17)), into the right hand side of (9), dividing across by $\mathsf{P}(Z \le l)$, and using the definition of $\mathsf{E}_l(\cdot)$, we have,

$$\log\left(\frac{\pi + \mathsf{G}_l(Z)}{\mathsf{P}(Z \le l)}\right) - \mathsf{E}_l(\log(Z)) - \frac{\eta\pi}{\pi + \mathsf{G}_l(Z)} = 0$$

$$\log\left(\frac{\pi}{\mathsf{P}(Z \le l)} + \mathsf{E}_l(Z)\right) - \mathsf{E}_l(\log(Z)) - \frac{\eta\pi}{\pi + \mathsf{G}_l(Z)} = 0$$

$$\log\left[\left(\frac{\pi}{\mathsf{G}_l(Z)} + 1\right)\mathsf{E}_l(Z)\right] + \log\left(e^{-\mathsf{E}_l(\log(Z))}\right)$$
$$-\frac{\eta\pi}{\pi + \mathsf{G}_l(Z)} = 0$$

Rearranging terms, we get,

$$\log\left(\frac{\pi + \mathsf{G}_l(Z)}{\mathsf{G}_l(Z)}\right) + \log\left(\mathsf{E}_l(Z)e^{-\mathsf{E}_l(\log(Z))}\right)$$
$$-\frac{\eta\pi}{\pi + \mathsf{G}_l(Z)} = 0$$

Denote $b_l := \log\left(\mathsf{E}_l(Z)e^{-\mathsf{E}_l(\log(Z))}\right)$. Then, we have,

$$\log\left(\frac{\pi + \mathsf{G}_l(Z)}{\mathsf{G}_l(Z)}\right) + b_l - \frac{\eta\pi}{\pi + \mathsf{G}_l(Z)} = 0$$

Using (17), we have

$$\frac{\mathsf{G}_l(Z)}{\pi + \mathsf{G}_l(Z)} = \frac{\mathsf{G}_l(Z)}{l\mathsf{P}(Z \le l)} = \frac{\mathsf{E}_l(Z)}{l}$$

which, with the previous equation, yields

$$\log\left(\frac{l}{\mathsf{E}_l(Z)}\right) + b_l - \eta\left(1 - \frac{\mathsf{E}_l(Z)}{l}\right) = 0$$

Recall that $l$ is actually $l(\pi)$. We now find that $\pi$ appears in the equation only as $l(\pi)$. Hence we can view this as an equation in the variable $l(= \frac{1}{\lambda})$. Rearranging terms, we get

$$-\log\left(\frac{\mathsf{E}_l(Z)}{l}\right) + \eta\frac{\mathsf{E}_l(Z)}{l} = -(b_l - \eta)$$

Exponentiating both sides, and substituting back for $b_l$, yields

$$\frac{\mathsf{E}_l(Z)}{l}e^{-\eta\frac{\mathsf{E}_l(Z)}{l}} = \mathsf{E}_l(Z)e^{-\mathsf{E}_l(\log(Z))}e^{-\eta}$$

On cancelling $\mathsf{E}_l(Z)$, and transposing terms, we next obtain

$$e^{-\eta\left(\frac{\mathsf{E}_l(Z)}{l} - 1\right)} = e^{-\mathsf{E}_l\left(\log\left(\frac{Z}{l}\right)\right)}$$

or,

$$e^{-\eta\left(\mathsf{E}_l\left(\frac{Z-l}{l}\right)\right)} = e^{-\mathsf{E}_l\left(\log\left(\frac{Z}{l}\right)\right)}$$

Taking log on both sides, we have,

$$\eta\mathsf{E}_l\left(\frac{Z-l}{l}\right) = \mathsf{E}_l\left(\log\left(\frac{Z}{l}\right)\right)$$

In terms of $G_l(\cdot)$, this is equivalent to

$$\eta G_l\left(\frac{Z-l}{l}\right) = G_l\left(\log\left(\frac{Z}{l}\right)\right)$$

which is the desired result after writing $l = \frac{1}{\lambda}$. ∎

We next address the question of a *unique* positive solution of (15). The following lemma guarantees the existence of a unique positive solution, when $f(\cdot)$, the density of $\frac{\alpha H}{\sigma^2}$, satisfies a certain monotonicity condition.

*Lemma A.2:* (15) has at most one positive solution if for any $\lambda_1 > \lambda_2 > 0$, $\frac{f\left(\frac{\lambda_2}{y}\right)}{f\left(\frac{\lambda_1}{y}\right)}$ is a strictly monotone decreasing function of $y$.

*Proof:* Expanding $G_{\frac{1}{\lambda}}(\cdot)$, (15) becomes,

$$\eta \int_0^{\frac{1}{\lambda}} (\lambda z - 1)g(z)dz - \int_0^{\frac{1}{\lambda}} \log(\lambda z)g(z)dz = 0$$

Rewriting the equation in terms of $f(\cdot)$, we have,

$$\int_0^{\frac{1}{\lambda}} (\eta(\lambda z - 1) - \log(\lambda z))\frac{1}{z^2}f\left(\frac{1}{z}\right)dz = 0$$

In this last equation change the variable to $y := \lambda z$, yielding

$$\int_0^1 (\log(y) - \eta(y-1))\frac{\lambda^2}{y^2}f\left(\frac{\lambda}{y}\right)dy = 0 \qquad (18)$$

Define $c(y) := (\log(y) - \eta(y-1))\frac{1}{y^2}$ and $b_\lambda(y) := f\left(\frac{\lambda}{y}\right)$. Thus, we are interested in a positive $\lambda$ that solves

$$\int_0^1 c(y)b_\lambda(y)dy = 0$$

Observe that $\lim_{y\to 0} c(y) = -\infty$ and $c(1) = 0$. Further, there exists a unique $y'$ such that $c(y) \le 0$ for all $0 \le y \le y'$ and $c(y) \ge 0$ for all $y' \le y \le 1$. Since $b_\lambda(y) \ge 0$ for all $y$ and $\lambda$, we have $c(y)b_\lambda(y) \le 0$ for all $0 \le y \le y'$ and $c(y)b_\lambda(y) \ge 0$ for all $y' \le y \le 1$.

Consider $\lambda_1, \lambda_2$ such that $\lambda_1 > \lambda_2 > 0$. By hypothesis, $\frac{b_{\lambda_2}(y)}{b_{\lambda_1}(y)}$ is a strictly monotone decreasing function of $y$. Hence, $\frac{c(y)b_{\lambda_2}(y)}{c(y)b_{\lambda_1}(y)}$ is also a strictly monotone decreasing function of $y$. We then have,

$$\frac{\int_0^{y'} |c(y)|b_{\lambda_2}(y)dy}{\int_0^{y'} |c(y)|b_{\lambda_1}(y)dy} = \frac{\int_0^{y'} |c(y)|\frac{b_{\lambda_2}(y)}{b_{\lambda_1}(y)}b_{\lambda_1}(y)dy}{\int_0^{y'} |c(y)|b_{\lambda_1}(y)dy}$$

$$> \frac{b_{\lambda_2}(y')}{b_{\lambda_1}(y')},$$

And,

$$\frac{\int_{y'}^1 c(y)b_{\lambda_2}(y)dy}{\int_{y'}^1 c(y)b_{\lambda_1}(y)dy} = \frac{\int_{y'}^1 c(y)\frac{b_{\lambda_2}(y)}{b_{\lambda_1}(y)}b_{\lambda_1}(y)dy}{\int_{y'}^1 c(y)b_{\lambda_1}(y)dy}$$

$$< \frac{b_{\lambda_2}(y')}{b_{\lambda_1}(y')}$$

Hence,

$$\frac{\int_0^{y'} |c(y)|b_{\lambda_2}(y)dy}{\int_0^{y'} |c(y)|b_{\lambda_1}(y)dy} > \frac{\int_{y'}^1 c(y)b_{\lambda_2}(y)dy}{\int_{y'}^1 c(y)b_{\lambda_1}(y)dy}$$

i.e.,

$$\frac{\int_0^{y'} |c(y)|b_{\lambda_2}(y)dy}{\int_{y'}^1 c(y)b_{\lambda_2}(y)dy} > \frac{\int_0^{y'} |c(y)|b_{\lambda_1}(y)dy}{\int_{y'}^1 c(y)b_{\lambda_1}(y)dy}$$

i.e., the ratio of the negative area of the integral to the positive area of the integral is a strictly monotonic function of $\lambda$. Hence, as $\lambda$ decreases, the integral can cross 0 at most once, or, there exists at most one (non-trivial) solution for (18). ∎