On the Limits of Spatial Reuse and Cooperative Communication for Dense Wireless Networks

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***Abstract*— We consider a dense ad hoc wireless network comprising n nodes confined to a given two dimensional region of fixed area. For the Gupta-Kumar ([1]) random traffic model and a realistic interference and path loss model (i.e., the channel power gains are bounded above by 1, and are bounded below by a strictly positive number), we study the scaling of the aggregate end-to-end throughput with respect to the network average power constraint, $\bar{P}$, and the number of nodes, n. The network power constraint $\bar{P}$ is related to the per node power constraint,p, as $\bar{P} = np$. For large $\bar{P}$, we show that the throughput saturates as $\Theta(\log(\bar{P}))$, irrespective of the number of nodes in the network. For moderate $\bar{P}$, which can accommodate spatial reuse to improve end-to-end throughput, we observe that the amount of spatial reuse feasible in the network is limited by the diameter of the network. In fact, we observe that the end-to-end path loss in the network and the amount of spatial reuse feasible in the network are inversely proportional. This puts a restriction on the gains achievable using the cooperative communication techniques studied in [2] and [3], as these rely on direct long distance communication over the network.**

I. Introduction

We consider a wireless network comprising n nodes confined to a given two dimensional region of fixed area A. Such networks are called dense or fixed SNR networks, because, the attenuation between any transmitter-receiver pair is lower bounded by a positive quantity independent of n. Source-destination (s-d) pairs are chosen randomly (as in the Gupta-Kumar random traffic model, see [1]) and the s-d pairs communicate by sharing the common wireless channel. For an average power constraint p at a node, the total network average power constraint, $\bar{P}$, is given by $\bar{P} = np$. For a realistic interference and path loss model, we study the scaling of the aggregate end-to-end throughput between the s-d pairs with respect to the network power constraint $\bar{P}$, and the number of nodes n.

Using a far-field path loss model of $\frac{1}{d^\eta}$ for every transmitter-receiver separation of d, Gupta and Kumar ([1]) showed that the end-to-end throughput of dense wireless networks scales as $\Theta\left(n^{\frac{1}{2}}\right)$. It was observed in [4] that, $\Theta\left(n^{\frac{1}{2}}\right)$ scaling is not feasible in realistic scenarios, as the far-field path loss model (used in [1]) provides a channel power gain greater than unity for very small d. In our work, we note that the scaling laws of dense wireless networks (with a realistic path loss model) depend not only on the number of nodes (n), but also on the network power constraint ($\bar{P}$). Our main result is that the end-to-end throughput of dense networks scales only as $\Theta(\log(\bar{P}))$ (or as $\Theta(\log(n))$, when $\bar{P} = np$ for a fixed p), due to interference from simultaneous transmitters and bounded distance between any transmitter-receiver pair. This contrasts with the $\Theta\left(n^{\frac{1}{2}}\right)$ scaling achievable for an extended network, where the size of the network scales as n (see for e.g., [1] and [3]). Viewed differently, the logarithmic scaling of the aggregate end-to-end throughput follows from the fact that the maximum achievable bit-rate in the network scales only as $\Theta(\log(\bar{P}))$ or $\Theta(\log(n))$, and not as $\Theta(n)$ (as in extended networks).

The logarithmic scaling, for n tending to infinity, or, for very large $\bar{P}$, is achieved using direct communication between the source-destination pairs, without any spatial reuse. However, better scaling results are achievable for small and moderate $\bar{P}$, by using spatial reuse, multihopping or other communication techniques. For the path loss model of $\frac{1}{d^\eta}$ for any transmitter-receiver pair separated by a distance d, [1] showed that spatial reuse and multihopping achieves an end-to-end throughput of $\Theta\left(n^{\frac{1}{2}}\right)$. A recent result, [2], achieved $\Theta\left(n^{\frac{2}{3}}\right)$ throughput using cooperative communication techniques, for a rich scattering environment. Using a similar cooperative communication technique (as in [2]) and by implementing a hierarchy, [3] obtained a $\Theta(n)$ throughput for dense wireless networks. The above results (as reported in [1], [2] and [3]) are not feasible for a realistic path loss scenario, and the scaling fails when the nodes become sufficiently close. While it is true that the scaling does not hold for n tending to infinity, we are interested in understanding the feasibility of the scaling laws for sufficiently large n (when the path loss model of $\frac{1}{d^\eta}$ holds). For such a scenario (when the path loss model holds for the given area A and a node density n), we observe that the amount of spatial reuse feasible in the network is limited by the diameter of the network. In fact, we show that the spatial reuse achievable in the network is inversely proportional to the end-to-end path loss in the network. This puts a restriction on the gains achievable using cooperative communication techniques discussed in [2] and [3], as they rely on direct

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communication over long distances in the network. We observe that, while spatial reuse and multihopping (as reported in [1]) can provide throughput enhancements for sufficiently large n, even in realistic scenarios, cooperative communication gains (as reported in [2] and [3]) may not be achievable.

Outline of the Paper : In Section II, we define the dense wireless network model, the realistic interference and path loss model and the objective function. In Section III, we show that the aggregate throughput of a dense network scales only as $\Theta(\log(\bar{P}))$ or $\Theta(\log(n))$. We discuss the feasibility of spatial reuse and cooperative communication for practical wireless networks in Section IV. We finally conclude the paper in Section V.

II. Network Model

We consider a wireless network comprising n nodes, distributed uniformly over a two dimensional region of fixed area A.

- $\frac{n}{2}$ source-destination pairs are formed in the network, with each node belonging to a distinct s-d pair. The s-d pairs are chosen randomly such that the mean s-d pair distance is $O(1)$, with respect to the diameter of the network.
- The s-d pairs communicate by sharing the common wireless channel. The gain between any transmitter-receiver pair is assumed fixed, and determined by the path loss that has a power law depending on the path length.
- We consider a total network average power constraint $\bar{P}$, accounting only for the transmit power of all the nodes in the network. Further, the nodes have an individual average power constraint p, that is related to $\bar{P}$ as $\bar{P} = np$. In our work, we assume that p is fixed for a given scenario, and hence, the network power constraint $\bar{P}$ scales as n. We do not model a maximum power constraint per node in the model.
- We assume that the system is slotted and nodes communicate over slots of fixed duration. When the nodes use single user decoding transceivers, we assume that the bit rate achieved between a transmitter and receiver is given by Shannon's formula, $C = \log_2(1 + \text{SINR})$ bits per symbol. Further, when the nodes communicate cooperatively, we assume that nodes are synchronised without any additional overheads.

A. Interference and Path loss Model

We consider a realistic physical model of interference (SINR based) in the network. In [1], it was assumed that the power gain between a transmitter and a receiver scaled with the distance d as $\frac{1}{d^\eta}$, where $\eta > 2$ is the path loss exponent. While this holds true for far-field distances, the above model is not appropriate when the receiver is very close to the transmitter. In our work, we use a generalized model in which the channel power gain between (i, j) is $\alpha_{i,j}$, where $0 < \alpha_A \leq \alpha_{i,j} \leq 1$. α_A is the minimum channel power gain between any transmitter-receiver pair, and is related to the diameter of the network, d_A, as $\alpha_A = \frac{1}{d_A^\eta}$. The assumption $\alpha_{i,j} \leq 1$ implies that a receiver cannot receive power more than the power transmitted.

B. Objective

Our objective is to study the scaling of the aggregate end-to-end throughput of the described wireless network for the interference and path loss model discussed above. We study the scaling laws for different network power constraint regimes - large $\bar{P}$ (in terms of $\bar{P}$) and moderate $\bar{P}$ (in terms of n). We consider spatial reuse, multihopping and cooperative communication as the strategies used in the network.

III. Scaling Laws for Large $\bar{P}$

In this section, we will obtain scaling laws of dense wireless networks with respect to $\bar{P}$. Suppose that the source-destination pairs are chosen arbitrarily (instead of randomly, as stated earlier), such that the s-d pairs are chosen as close as possible. Then, the aggregate end-to-end throughput achievable in this scenario is the same as the maximum spatial reuse feasible. Clearly, the throughput achieved in this scenario, upper bounds the throughput achieved for the random traffic model (notice that the first phase in every communication strategy studied in [1], [2] and [3] is spatial reuse). Now, we assume that the nodes use single user decoding receivers, treating every simultaneous transmission (other than the intended one) as interference. We will now upper bound the bit rate achievable in this scenario.

A. An upper bound on the Network Throughput

Consider a slot t, when node $i, 1 \leq i \leq n$, transmits with power $P_i(t)$, and the transmit powers are such that they satisfy a network power constraint, $\sum_{i=1}^{n} P_i(t) \leq \bar{P}(t)$. For ease of notation, we will omit the index t now, and include it again later (at the end of this section). The SINR achievable (in slot t) at the receiver of a transmitter i is bounded above by

$$SINR \leq \frac{\alpha_i P_i}{N + \sum_{\{j \neq i\}} \alpha_j P_j}$$

where α_i and α_j are the constant gains at the receiver from the transmitters i and j and N is the noise power. Then, it follows from the interference model, that the SINR is bounded above as

$$\text{SINR} \leq \frac{P_i}{N + \alpha_A \sum_{\{j \neq i\}} P_j}$$

For an allocated total network power of $\bar{P}$, an optimal power allocation (that maximizes throughput) must satisfy $\sum_i P_i = \bar{P}$. Hence, using the equality $\sum_i P_i = \bar{P}$ in the above expression, we have,

$$SINR \leq \frac{P_i}{N + \alpha_A(\bar{P} - P_i)}$$

Now, the maximum throughput achievable in the network is bounded above by

$$C(\alpha_A) := \sum_{i \in \mathcal{T}} \log \left(1 + \frac{P_i}{N + \alpha_A(\bar{P} - P_i)}\right)$$

where $\mathcal{T}$ indexes the set of transmitters with positive power. We denote the above expression as $C(\alpha_A)$, denoting the dependence on the parameter α_A. We will now obtain an upper bound for $C(\alpha_A)$ by optimising the above expression for P_i, i.e., we will maximize $C(\alpha_A)$ subject to the power constraint $\sum_i P_i \leq \bar{P}$.

B. Optimization Problem

Define $f(P) := \log\left(1 + \frac{P}{N+\alpha_A(\bar{P}-P)}\right)$. Then the optimization problem can be written as

$$\max \sum_i f(P_i) \tag{1}$$

subject to the power constraint

$$\sum_i P_i \leq \bar{P}$$

Lemma 3.1: $f(P)$ is monotone increasing with P for $0 \leq P \leq \bar{P}$. ■

Lemma 3.2: Suppose that $2\alpha_A - 1 > 0$. Then, f is convex in P for $0 \leq P \leq \bar{P}$. Further, the solution for the optimization problem in (1) is to allot all the power to a single transmitter. And the optimal value for the objective function in (1) is $f(\bar{P})$.

Proof: See Appendix A for the proof. ■

Lemma 3.3: Suppose that $2\alpha_A - 1 < 0$. For large $\bar{P}$, there exists a P', $0 \leq P' \leq \bar{P}$, such that $f(P)$ is concave upto P' and convex thereafter.

Proof: See Appendix A for the proof. ■

$f(\bar{P}) = \log\left(1 + \frac{\bar{P}}{N}\right)$, increases to infinity with $\bar{P}$, and $f'(0) = \frac{1}{N+\alpha_A\bar{P}}$, decreases to 0 as $\bar{P}$ increases. Now, observe that, for large $\bar{P}$, we have $f'(0) \leq \frac{f(\bar{P})}{\bar{P}}$. The following lemma upper bounds $f(P)$ for all $0 \leq P \leq \bar{P}$.

Theorem 3.1: Suppose that $2\alpha_A - 1 < 0$. For large $\bar{P}$, $f(P) \leq \frac{f(\bar{P})}{\bar{P}} P$ for all $0 \leq P \leq \bar{P}$.

Proof: See Appendix A for the proof. ■

Define $g(P) := f(\bar{P})\frac{P}{\bar{P}}$. Let $\{\tilde{P}_i\}$ be an optimal solution for the optimization problem (1). From Theorem 3.1 and the definition of $g(\cdot)$, we have,

$$\sum_i f(\tilde{P}_i) \leq \sum_i g(\tilde{P}_i)$$

whenever $\bar{P}$ is large enough. Observe that,

$$\sum_i g(\tilde{P}_i) = \sum_i \frac{f(\bar{P})}{\bar{P}}\tilde{P}_i = \frac{f(\bar{P})}{\bar{P}}\sum_i \tilde{P}_i = \frac{f(\bar{P})}{\bar{P}}\bar{P} = f(\bar{P})$$

This implies that for large $\bar{P}$, $f(\bar{P}) \geq \sum_i f(P_i)$, for any power allocation, or $f(\bar{P})$ is the optimal solution. Thus, $\log\left(1 + \frac{\bar{P}}{N}\right)$ is an upper bound on the network throughput. This implies that the maximum achievable bit rate for a dense network with arbitrary s-d pairs is $O(\log(\bar{P}))$.

Remark:

1) We have only shown that for a per slot network power constraint $\bar{P}(t)$, the aggregate bit rate scales as $\log(\bar{P}(t))$. It is now straightforward to extend the above results to a sequence of network power constraints $\{\bar{P}(t), t = 1, 2, \cdots\}$ which satisfy $\lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t} \bar{P}(i) \leq \bar{P}$.
2) We have shown that the aggregate bit rate of an arbitrary network scales as $O(\log(\bar{P}))$ (this is also the maximum bit rate achievable in the network). Hence, the aggregate end-to-end throughput of a random network (as defined in Section II) can scale only as $O(\log(\bar{P}))$.
3) Also, observe that the above results depend only on α_A and $\bar{P}$, but are independent of the number of the nodes in the network (and hence, on the spacing between the nodes).
4) We call $\bar{P}$ greater than the threshold (according to Theorem 3.1) for which the logarithmic scaling holds, as large power regime. And networks with total power constraint $\bar{P}$ lesser than this threshold are called moderate power networks.
5) For an extended network, where the network size scales with n, the path loss from the farthest node decreases to 0. For $\eta > 2$, [1] showed that the cumulative interference from simultaneous transmitters can then be bounded, thus achieving $\Theta(n)$ aggregate bit rate with spatial reuse. Using multihopping strategy, [1] achieved $\Theta\left(n^{\frac{1}{2}}\right)$ end-to-end throughput for extended networks.
6) Consider a simple TDM scheme, where each node transmits with power $\bar{P} = np$, to its intended destination in its slot. Since a node gets access to the channel once in every n slots, the average power per node is $\frac{\bar{P}}{n} = p$. And the achieved bit rate in the proposed scheme scales as $\log(n)$. This proves the achievability of $\Theta(\log(n))$ for dense wireless networks.

The following theorem summarizes the above arguments.

Theorem 3.2: The aggregate end-to-end throughput of a dense wireless network scales as $\Theta(\log(\bar{P}))$, where $\bar{P}$ is the network average power constraint. In terms of the number of nodes, n, the maximum achievable throughput of a dense network scales as $\Theta(\log(n))$. ■

IV. Spatial Reuse and Cooperative Communication

For dense wireless networks, [1], [2] and [3] achieved $\Theta\left(n^{\frac{1}{2}}\right)$, $\Theta\left(n^{\frac{2}{3}}\right)$ and $\Theta(n)$ end-to-end throughput respectively, by using a far field path loss model (with a path loss of $\frac{1}{d^\eta}$ for any transmitter-receiver separation of d). As observed in [4], this model requires power amplification by the channel for sufficiently small values of d, and hence, is not practical. The scaling fails when the nodes become sufficiently close, i.e., when n tends to infinity. In this section, we are interested in understanding the feasibility of the communication strategies discussed in [1] (spatial reuse and multihopping), [2] (spatial reuse, multihopping and cooperative communication) and [3] (spatial reuse, multihopping, cooperative communication and hierarchy) for sufficiently large n, when the path loss model of $\frac{1}{d^\eta}$ still holds.

For example, consider a 1Km $\times$ 1Km planar area, with a million nodes arranged in a square grid with a minimum spacing of 1 metre between them. For a carrier frequency of

3 GHz, the carrier wavelength is around $0.1m$ much smaller than the node separation of 1m. The path loss model holds for this deployment, and hence, we could expect that such results as spatial reuse (of $\Theta(n)$), multihopping (of $\Theta\left(n^{\frac{1}{2}}\right)$) and cooperative communication (of $\Theta\left(n^{\frac{2}{3}}\right)$ or $\Theta(n)$) hold approximately (for e.g., with some probability). Observing that spatial reuse is essential to every communication strategy (studied in [1], [2] and [3]), in this context, we will study the feasibility of $\Theta(n)$ spatial reuse in the network, and the impact it has on using cooperative communication techniques.

The following simple calculations given below show that in order to support a spatial reuse of $\Theta(n)$, the network size must be at least as large as $\Theta\left(n^{\frac{1}{\eta}}\right)$. Let us fix the SINR requirement for point-to-point communication to β, independent of the number of nodes and the dimensions of the network. Suppose that all the transmissions involve the constant transmit power p. Let $S(n, A)$ denote the spatial reuse achievable in the network with n nodes while supporting a SINR of β. Then, this implies that the maximum interference gain observed at any receiver needs to be bounded above, as seen below.

$$\beta \leq \text{SINR} \leq \left(\frac{p}{N + p(S(n, A) - 1)\alpha_A}\right)$$

This implies that

$$(S(n, A) - 1)\alpha_A \leq \gamma$$

for some constant, γ, independent of n or p. Absorbing the constants and simplifying the expression, we have,

$$S(n, A)\alpha_A \leq 1$$

Hence, to achieve a spatial reuse of $\Theta(n)$, we require,

$$\Theta(n)\alpha_A \leq 1 \tag{2}$$

In terms of d_A, we have,

$$\Theta(n)\frac{1}{d_A^\eta} \leq 1$$

or,

$$\Theta(n) \leq d_A^\eta$$

The above expression implies that the total spatial reuse feasible in the network is bounded by the dimensions of the network. In other words, to support a spatial reuse of $\Theta(n)$, the dimensions of the network should at least scale as $\Theta\left(n^{\frac{1}{\eta}}\right)$, or the area A should be as large as $\Theta\left(n^{\frac{2}{\eta}}\right)$.

Viewed differently, the end-to-end path loss between any source-destination pair (according to the random traffic model) will scale at least as $\Theta\left(\frac{1}{n}\right)$ (from Equation (2)), when the spatial reuse scales as $\Theta(n)$. The cooperative communication models described in [2] and [3], require the source cluster (the biggest cluster containing the source node) and the destination cluster (the biggest cluster containing the destination node) to communicate directly, using cooperative communication involving the nodes in the cluster. For a spatial reuse of $\Theta(n)$, and the number of nodes in the cluster M ($M < \Theta(n)$, for e.g., in [2], $M = n^{\frac{2}{3}}$), an upper bound on the maximum bit rate achievable in the cooperative communication phase is,

$$n \log\left(1 + \frac{p\alpha_A}{N}\right) \leq n \log\left(1 + \frac{p}{\Theta(n)N}\right)$$

where we have modeled the cooperative communication phase as comprising of n parallel independent channels, with only the path loss between a transmitter-receiver pair in the cluster. Observe that the throughput, as given by the above expression does not scale as $\Theta(n)$. In fact, it is bounded above by a constant. The key observation is that the path loss that permits spatial reuse in the channel is so restrictive that it is unable to support long distance MIMO communications. It is easy to verify from the above formulation that, for any power allocation with a total network power of np, the achievable throughput using cooperative communication (as reported in [2] and [3]) is bounded by a constant, independent of n.

Returning to the example of a million nodes arranged in a grid in a 1Km × 1Km planar area, we see that, while spatial reuse and multihopping may increase the performance of the system (as compared to direct transmissions involving the s-d pairs), cooperative communication does not. In other words, while spatial reuse and multihopping can coexist, even in a practical scenario, for sufficiently large n, spatial reuse and cooperative communication (as reported in [2] and [3]) cannot. The direct communication between the source-destination clusters should be avoided in order to enhance the throughput in realistic scenarios. However, by restricting the communication distance between the clusters, we lose throughput due to multihopping costs.

V. Conclusions

The important feature of a dense network, as compared to an extended network is the positive interference due to a simultaneous transmission any where in the network. We have observed that this implies that the scaling results are a function of both the number of nodes and the network power. More specifically, for large power networks, we observe that the achievable throughput scales only as $\Theta(\log(\bar{P}))$, irrespective of the number of nodes in the network. However, for moderate $\bar{P}$, when spatial reuse may be efficient, we showed that spatial reuse puts a restriction on the network size, which affects the gains achieveable using cooperative communication techniques.

VI. Acknowledgements

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APPENDIX

A. Proofs for Section III

Lemma 3.1 : $f(P)$ is monotone increasing in P for $0 \leq P \leq \bar{P}$.

Proof: Differentiating $f(P)$ with respect to P, we have,

$$\begin{aligned} f'(P) &= \frac{1}{1+\frac{P}{N+\alpha_A\bar{P}-\alpha_A P}} \\ &\times \left(\frac{1}{N+\alpha_A\bar{P}-\alpha_A P} + \frac{\alpha_A P}{(N+\alpha_A\bar{P}-\alpha_A P)^2}\right) \\ &= \frac{1}{N+\alpha_A\bar{P}-\alpha_A P+P}\left(1+\frac{\alpha_A P}{(N+\alpha_A\bar{P}-\alpha_A P)}\right) \\ &= \left(\frac{1}{N+\alpha_A\bar{P}+P-\alpha_A P}\right)\left(\frac{(N+\alpha_A\bar{P})}{N+\alpha_A\bar{P}-\alpha_A P}\right) \end{aligned}$$

Clearly, $f'(P) \geq 0$ for all $0 \leq P \leq \bar{P}$. Hence, $f(P)$ is monotone increasing for all $0 \leq P \leq \bar{P}$. ■

Differentiating $f'(P)$ with respect to P again, we have,

$$f''(P) = \frac{d}{dP}\left(\frac{1}{K+(1-\alpha_A)P}\frac{1}{K-\alpha_A P}\right)$$

We have used the substitution $K := N + \alpha_A\bar{P}$ in the above expression. Also, we are interested only in the sign of $f''(P)$, hence, we have ignored $(N+\alpha_A\bar{P})$ in the numerator as well.

$$\begin{aligned} f''(P) &= \frac{1}{K-\alpha_A P}\left(\frac{-1(1-\alpha_A)}{(K+(1-\alpha_A)P)^2}\right) \\ &+ \frac{1}{K+(1-\alpha_A)P}\left(\frac{-1\times-\alpha_A}{(K-\alpha_A P)^2}\right) \\ &= \frac{1}{(K-\alpha_A P)(K+(1-\alpha_A)P)} \\ &\times \left(\frac{-(1-\alpha_A)}{K+(1-\alpha_A)P}+\frac{\alpha_A}{K-\alpha_A P}\right) \end{aligned}$$

Clearly, $K - \alpha_A P = N + \bar{P} - \alpha_A P \geq 0$ (for $0 \leq P \leq \bar{P}$) and $K + (1-\alpha_A)P \geq 0$. Hence, we will concentrate only on the terms inside the braces,

$$\begin{aligned} f''(P) &= \frac{-(1-\alpha_A)}{K+(1-\alpha_A)P}+\frac{\alpha_A}{K-\alpha_A P} \\ &= \frac{-(K-\alpha_A P)(1-\alpha_A)+\alpha_A(K+(1-\alpha_A)P)}{(K+(1-\alpha_A)P)(K-\alpha_A P)} \end{aligned}$$

Ignoring the denominator (which is always positive), we have,

$$\begin{aligned} f''(P) &= -[K-K\alpha_A+\alpha_A^2 P-\alpha_A P] \\ &+ [\alpha_A K+\alpha_A P-\alpha_A^2 P] \\ &= (2\alpha_A-1)K+2\alpha_A P(1-\alpha_A) \end{aligned}$$

Clearly, $1 \geq \alpha_A$. Now, if $2\alpha_A - 1 > 0$, then we see that the above expression is positive for all $0 \leq P \leq \bar{P}$. Hence, $f'' \geq 0$, or, the function $f(P)$ is convex increasing.

Lemma 3.2 : Suppose $2\alpha_A - 1 > 0$. Then, f is convex in P for $0 \leq P \leq \bar{P}$. Further, the solution for the optimization problem in (1) is to allot all the power to a single transmitter. And the optimal value for (1) is $f(\bar{P})$. ■

Now suppose that $2\alpha_A - 1 < 0$. Then, $f''(0) < 0$ and for large enough $\bar{P}$, $f''(\bar{P}) > 0$, i.e., for large $\bar{P}$, there would exist a $P' \leq \bar{P}$ such that $f''(P') \leq 0$ for all $0 \leq P \leq P'$ and $f''(P) \geq 0$ for all $P' \leq P \leq \bar{P}$. The following lemma summarizes the idea.

Lemma 3.3 : Suppose that $2\alpha_A - 1 < 0$. For large $\bar{P}$, there exists a P', $0 \leq P' \leq \bar{P}$, such that $f(P)$ is concave upto P' and convex there after. ■

From the definition of $f(\cdot)$, we see that, $f(\bar{P}) = \log\left(1+\frac{\bar{P}}{N}\right)$ increases to infinity with $\bar{P}$. Also, $f'(0) = \frac{1}{N+\alpha_A\bar{P}}$, decreases to 0 as $\bar{P}$ increases. Further, for large $\bar{P}$, we see that $f'(0) \leq \frac{f(\bar{P})}{\bar{P}}$.

Theorem 3.1 : Suppose that $2\alpha_A - 1 < 0$. For large $\bar{P}$, $f(P) \leq \frac{f(\bar{P})}{\bar{P}}P$.

Proof: From Lemma 3.3, we know that $f(P)$ is concave upto P'. Hence, $f(P) \leq f(0) + f'(0)P$ for $0 \leq P \leq P'$. Since $f(0) = 0$, we have, $f(P) \leq f'(0)P$. For large $\bar{P}$, we have $f'(0) \leq \frac{f(\bar{P})}{\bar{P}}$ (from the previous arguments). Hence, for $0 \leq P \leq P'$,

$$f(P) \leq \frac{f(\bar{P})}{\bar{P}}P$$

In the region $P' \leq P \leq \bar{P}$, $f(P)$ is convex increasing, hence, we have,

$$f(P) - f(P') \leq \frac{(f(\bar{P}) - f(P'))}{\bar{P}-P'}(P-P')$$

Simplifying the above expression, we have,

$$f(P) \leq f(P') + f(\bar{P})\frac{(P-P')}{\bar{P}-P'} - f(P')\frac{(P-P')}{\bar{P}-P'}$$

Or,

$$f(P) \leq f(\bar{P})\frac{(P-P')}{\bar{P}-P'} + f(P')\left(1-\frac{(P-P')}{\bar{P}-P'}\right)$$

Or,

$$f(P) \leq f(\bar{P})\frac{(P-P')}{\bar{P}-P'} + f(P')\frac{\bar{P}-P}{\bar{P}-P'}$$

Substituting $f(P') \leq f'(0)P'$, we have,

$$f(P) \leq f(\bar{P})\frac{(P-P')}{\bar{P}-P'} + f'(0)P'\frac{\bar{P}-P}{\bar{P}-P'}$$

For large $\bar{P}$, $f'(0) \leq \frac{f(\bar{P})}{\bar{P}}$. Substituting again, we get,

$$f(P) \leq f(\bar{P})\frac{(P-P')}{\bar{P}-P'} + \frac{f(\bar{P})}{\bar{P}}P'\frac{\bar{P}-P}{\bar{P}-P'}$$

Or,

$$f(P) \leq f(\bar{P})\left(\frac{(P-P')}{\bar{P}-P'} + \frac{P'}{\bar{P}}\frac{\bar{P}-P}{\bar{P}-P'}\right)$$

Simplifying the above expression, we have,

$$f(P) \leq f(\bar{P})\frac{P}{\bar{P}}$$

for all $P' \leq P \leq \bar{P}$, which completes the proof. ■