Aspects of Continuous- and Discrete-Time Signals and Systems

C.S. Ramalingam

Department of Electrical Engineering
IIT Madras

Scaling the Independent Axis

- Let $y(t) = x(at + b)$
- Can be done in two ways
  - Shift first and then scale:

$$\begin{aligned} w(t) &= x(t + b) \\ y(t) &= w(at) \\ &= x(at + b) \end{aligned}$$

  - Scale first and then shift:

$$\begin{aligned} w(t) &= x(at) \\ y(t) &= w(t + b/a) \qquad \text{shift by } b/a\text{, not } b \\ &= x[a(t + b/a)] \\ &= x(at + b) \end{aligned}$$

An Aspect of Scaling in Discrete-Time

- Consider $y[n] = x[n/2]$
- $y[n]$ is defined for even values of n only
- Wrong: $y[\text{odd}] = 0$

  Right: $y[\text{odd}] = \text{undefined}$
- Usual to set $y[\text{odd}] = 0$
  - but the above does not follow automatically from original definition
- For a discrete-time signal, $y[a] = \text{undefined}$ if $a \notin \mathbb{Z}$

Scaling Need Not Be Affine Only

- Consider $x(t) = 1$ for $0 < t \leq 1$
- What is $y(t) = x(e^t)$?
- Mellin Transform:

$$
\begin{aligned}
X_M(s) &= \int_0^\infty x(t)\, t^{s-1}\, dt \\
&= \underbrace{\int_{-\infty}^{\infty} x(e^{-t})\, e^{-st}\, dt}_{\text{Laplace transform of } x(e^{-t})}
\end{aligned}
$$

Periodic Signals

- $\exp(j\omega_0 n)$ is periodic with period N only if $\omega_0/2\pi = k/N$
  - $\exp(j\Omega_0 t)$ is periodic for *any* Ω_0 with period $T = 2\pi/\Omega_0$
- $\exp(j\omega_0 n)$ and $\exp(-j\omega_0 n)$ are two distinct exponentials
  - Their frequency content is the "same" but *one cannot be expressed as a (complex) constant times the other*
- Harmonics in the discrete-time case may "oscillate more rapidly" but their fundamental periods need not be different
  - $x_k[n] = \cos(2\pi nk/11)$: All x_k's have the same period, i.e., 11
  - This is not so for its continuous-time counterpart

Sinusoid With Time-Varying Frequency

- If the frequency of a sinusoid is constant, i.e., Ω_0, the signal is $x(t) = \sin(\Omega_0 t)$
- Consider *time-varying* frequency i.e, $\Omega(t)$
- Is it correct to write $x(t) = \sin(\Omega(t) \cdot t)$?
  - Wrong! i.e., in general, the above is not correct
- $x(t) = \sin(\phi(t))$, where $\phi(t) = \displaystyle\int_{-\infty}^{t} \Omega(\tau)\, d\tau$
- MATLAB:

```
>> phi = cumsum(omega);
>> x = sin(phi);
```

Impulse Response

- An LTI system has to have its initial conditions set to zero before exciting by an impulse to obtain $h(t)$ (or $h[n]$)
  - Otherwise the impulse response won't be unique
- All systems—both linear and non-linear have impulse response
  - In the case of an LTI system, h completely describes the system
  - For nonlinear systems h is not that useful

The Impulse Function

- An impulse is *not a function in the usual sense*
- Definition:

$$\delta(t) = 0 \qquad t \neq 0 \tag{1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{2}$$

- Wrong: $\delta(0) = \infty$
- Consider $x_\Delta(t) = \frac{1}{\Delta}$ for $\Delta \leq t \leq 2\Delta$ (where $\Delta > 0$)
- In the limit

$$\begin{aligned} x(t) &= \lim_{\Delta \to 0} x_\Delta(t) \\ &= \delta(t) \end{aligned}$$

The Impulse Function

- $x_\Delta(0) = 0$ for all values of Δ
  - Hence, in the limit, $x(0) = 0$
- To show $x(t) = 0$ for $t \neq 0$:
  - For *any* $t_0 > 0$, however small, there always exists a small enough value of Δ such that $x_\Delta(t_0) = 0$

$$\forall\ t_0 > 0,\ \exists\ \Delta_0 \text{ s.t. } \forall\ \Delta < \Delta_0,\ \ x_\Delta(t_0) = 0$$

  - Hence, $\delta(t_0) = \lim\limits_{\Delta \to 0} x_\Delta(t_0) = 0$
  - Since t_0 is arbitrary, $\delta(t) = 0$ for $t \neq 0$
- A function defined by (1) and (2) is not unique
  - $\delta(t)$ should be called as a functional or generalized function

The Impulse Function

- Be extremely careful when dealing with delta functions
  - $\delta(t)$ is actually an abbreviation for a limiting operation
- $\delta(t)$ is like a live wire!
  - Inside an integral, they are well-behaved: safe to use them
  - Bare $\delta(t)$ can give erroneous results if not carefully used
- Products or quotients of generalized functions not defined
  - Consider $\delta(t) = \delta(t) * \delta(t) = \int_{-\infty}^{\infty} \delta(\tau)\,\delta(t-\tau)\,d\tau$
  - At $t = 0$ is there a contradiction?
- Because area is unity, the zero width implies infinite height
  - Conventionally, height is proportional to area

Difference Between Continuous- and Discrete-Time Impulses

- Discrete-time impulse:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

- Perfectly well-behaved function, unlike its continuous-time counterpart
- Under scaling, these two functions behave very differently:
  - $\delta(at) = \dfrac{1}{|a|}\delta(t)$
  - $\delta[an] = \delta[n]$

Unit Step and Sinc Functions

- $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$
- $u(0) =$ undefined
  - $u(0)$ can be defined to be 0, 1, or any number
- There are *two types* of sinc functions:
  - Analog Sinc: $\dfrac{\sin(\pi\,\Omega)}{\pi\,\Omega}$

    CTFT of (CT) rectangular window; aperiodic
  - Digital Sinc: $\dfrac{\sin(N\,\omega/2)}{\sin(\omega/2)}$

    DTFT of (DT) rectangular window; periodic

    a.k.a Dirichlet kernel (`diric` function in MATLAB)

Convolution

- See Java applets at `http://www.jhu.edu/~signals`
- The "$*$" symbol is just notation

$$\begin{aligned} y(t) &= x(t) * h(t) \\ y(ct) &= c\, x(ct) * h(ct) \qquad \text{(prove this!)} \\ &\neq x(ct) * h(ct) \end{aligned}$$

- Convolution is a "smoothing" operation
- Apply the eigensignal $\exp(j\Omega t)$ to an LTI system with impulse response $h(t)$
- Output is $H(\Omega) \cdot \exp(j\Omega t)$ (reminiscent of $\mathbf{Ax} = \lambda \mathbf{x}$)
  - $H(\Omega)$ is the eigenvalue corresponding to $\exp(j\Omega t)$
  - This eigenvalue is nothing but the Fourier transform of $h(t)$

Discrete-Time Convolution

- The familiar one:

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

- Leave the first signal $x_1[k]$ unchanged
- For $x_2[k]$:
  - Flip the signal: k becomes $-k$, giving $x_2[-k]$
  - Shift the *flipped* signal to the *right* by n

    samples:
    k becomes $k - n$
    $x_2[-k] \rightarrow x_2[-(k-n)] = x_2[n-k]$
- Carry out sample-by-sample multiplication and sum the resulting sequence to get the output at time index n, i.e. $y[n]$

What happens to periodic signals?

- Suppose both signals are periodic (with same period)

$$
\begin{aligned}
x_1[n+N] &= x_1[n] \\
x_2[n+N] &= x_2[n]
\end{aligned}
$$

  Then $x_1[k]\, x_2[n_0 - k]$ will also be periodic (with period N)

- For each value of n_0 we get a different periodic signal (periodicity is N in all cases)

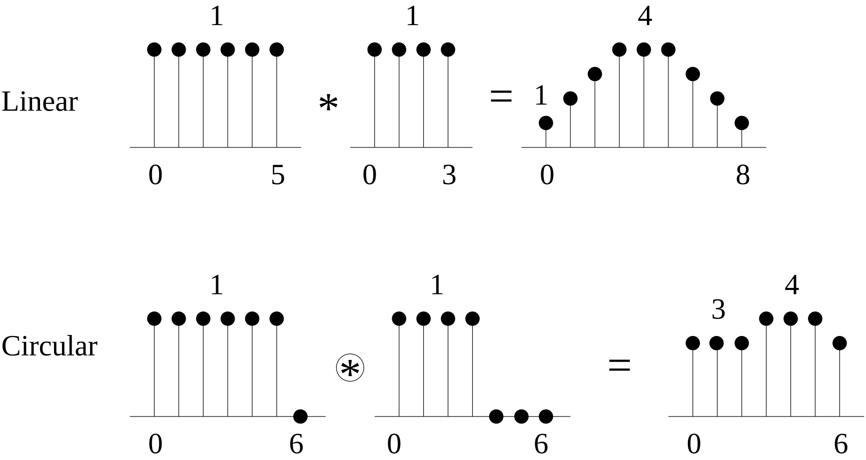
- $|y[n]|$ will be either 0 or ∞

$$y[n] \stackrel{?}{=} \sum_{k=0}^{N-1} \tilde{x}_1[k]\, \tilde{x}_2[n-k]$$

- $y[n]$ is periodic with period N
- $n-k$ can be replaced by $\langle n-k \rangle_N$ ("$n-k \bmod N$")
- "Circular" Convolution: $\tilde{y}[n] = \tilde{x}_1[n] \circledast \tilde{x}_2[n]$

$$\tilde{y}[n] \stackrel{\text{def}}{=} \sum_{k=0}^{N-1} \tilde{x}_1[k]\, \tilde{x}_2[\langle n-k \rangle_N] \qquad n = 0, 1, \ldots, N-1$$

Examples



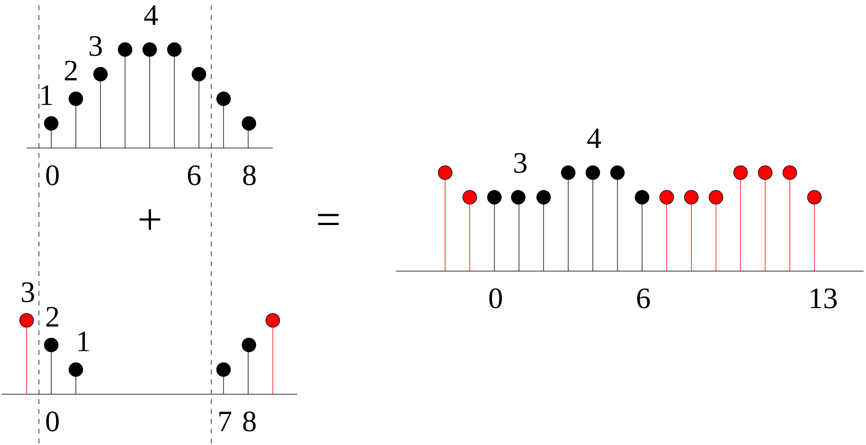
Relationship Between Linear and Circular Convolution

- If $x_1[n]$ has length P and $x_2[n]$ has length Q, then $x_1[n] \ * \ x_2[n]$ is $P + Q - 1$ long (e.g., $6 + 4 - 1 = 9$)

- $N \geq \max(P, Q)$. In general

$$\tilde{x}_1[n] \circledast \tilde{x}_2[n] \neq x_1[n] * x_2[n] \qquad n = 0, 1, \ldots, N-1$$

- Circular convolution can be thought of as repeating the result of linear convolution every N samples and adding the results (over one period)

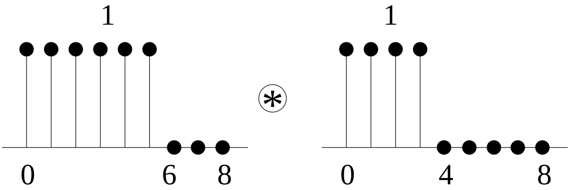


- But if $N \geq P + Q - 1$

$$\tilde{x}_1[n] \circledast \tilde{x}_2[n] = x_1[n] * x_2[n] \qquad n = 0, 1, \ldots, N-1$$

Linear Convolution via Circular Convolution

- If $N \geq 9$ one period of circular convolution will be equal to linear convolution.



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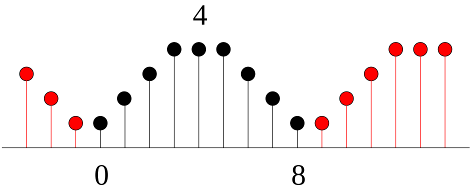
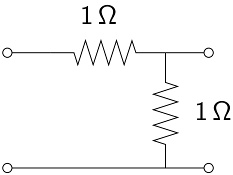


LTI Systems

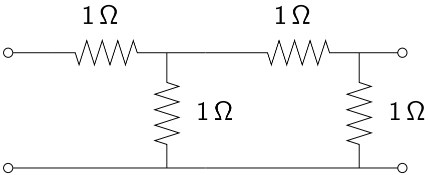
- Books differ in definition:
  - Oppenheim/Willsky:
    1. Initial conditions are not accessible
    2. If present, the system is defined to be quasi-linear
  - Lathi:
    1. Initial conditions are accessible
    2. Treated as separate sources $\Rightarrow$ system is still linear

       Not consistent with his black-box definition

- "Non-causal systems are not realizable"
  - True only if independent variable is time
  - In an image, "future" sample is either to the right or top of current pixel
-



$$\frac{v_o}{v_i} = \frac{1}{2}$$



$$\frac{v_o}{v_i} = \frac{1}{5} \neq \frac{1}{4}$$

  - Beware of loading! If the two sections are connected through a voltage-follower, overall transfer function will be $\frac{1}{4}$

Eigensignals of LTI Systems

- $\exp(j\Omega_0 t)$ is an eigensignal
  - So is $\exp(j\omega_0 n)$
- Is $\cos(\Omega_0 t)$ an eigensignal?
  - No!
  - If a certain condition is satisfied, $\cos(\Omega_0 t)$ can be an eigensignal. Derive this condition!
- Is $\exp(j\Omega_0 t)\, u(t)$ an eigensignal?
  - No!

LCCDE

- Networks containing only R, L, and C give rise to linear, constant-coefficient, differential equations
  - The DE coefficients are a function of R, L, and C, and network topology
  - If R, L, and C vary with time, the DE coefficients will also be a function of time $\Rightarrow$ linear time-varying system
- Maths approach: complementary function, particular integral
  - Complementary function: natural modes only
  - Particular integral: response due to forcing function
- EE approach: zero-input response, zero-state response
  - Zero-input response: natural modes only
  - Zero-state response: natural modes + response due to forcing function

- Particular Integral:
  - Depends only on the applied input
  - Does not contain any unknown constants
  - Sometimes misleadingly called "steady-state" response
    - What if the input is a decaying exponential ?
- Complementary Function
  - Independent of input, depends only on DE coefficients
  - CF of n-th order DE has n unknown constants $\Rightarrow$ need n auxiliary conditions to evaluate them
  - Auxiliary conditions are called "initial conditions" only if they are specified at $t = 0$

LCCDE

- To solve DE, we need auxiliary conditions, which are typically of the form $x(t_0)$, $x'(t_0)$, $x''(t_0)$, etc.
  - Typically, $t_0 = 0$ i.e., we are given initial conditions
- In circuit analysis, initial conditions are not given explicitly
- Instead, we are given capacitor voltages and inductor currents at $t = 0^-$
- From these we have to derive $x(0)$, $x'(0)$, etc. and then proceed to solve the DE

- Excitation is applied at $t = 0$. *In general*, the output will contain both natural response and forced response
- For stable systems, natural response will die out
  - Forced response also will die out if the input is not periodic
- Therefore, in certain applications, we should avoid the initial portion of the output
  - Coloured noise is obtained by passing white noise sequence through a (discrete-time) filter
  - The output can be considered stationary only if the initial transients are discarded

Resonance

- Resonance occurs even when a decaying input is applied
- Input: $x(t) = e^{-at}\, u(t)$
- Impulse response: $h(t) = e^{-at}\, u(t)$
- Output: $y(t) = t\, e^{-at}\, u(t)$

Time	Periodic	Non-Periodic
Continuous	Fourier Series	CT Fourier Transform
Discrete	DT Fourier Series (closely related to DFT)	DT Fourier Transform

- Notation for frequency:
    - Continous-time signal: F, Ω
    - Discrete-time signal: f, ω

Continuous-Time Fourier Series

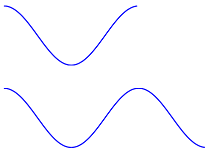
- The FS coefficients a_k can be plotted in two ways:

  (i) a_k vs. k (ii) a_k vs. Ω

- If the a_k's are plotted as a function of k, the plots will be identical for $x(t)$ and $x(ct)$
  - The actual frequency content cannot be determined if Ω_0 information is not available

- If the a_k's are plotted as a function of Ω, the scaling of the frequency axis will be clearly seen

- These two signals have very different FS coefficients!



- In general, there will be infinite number of harmonics

- Number of harmonics is finite
  - Equals N, where N is the periodicity
- Gibbs phenomenon does not exist in DTFS, since summation is finite
  - When all N terms are present, error is zero
- Closely related to the Discrete-Fourier Transform (DFT)
  - Efficient algorithm, called the Fast-Fourier Transform (FFT) exists for computing DFT coefficients

The Discrete Fourier Transform

- Given $x[n]$, $n = 0, 1, \ldots, N-1$ we define the DFT as

$$X[k] \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi nk/N}$$

- $X[k] = X[k+N]$, i.e., only N distinct values are present
- The inversion formula is

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j2\pi nk/N}$$

- $\tilde{x}[n] = x[n]$ for $n = 0, 1, \ldots, N-1$

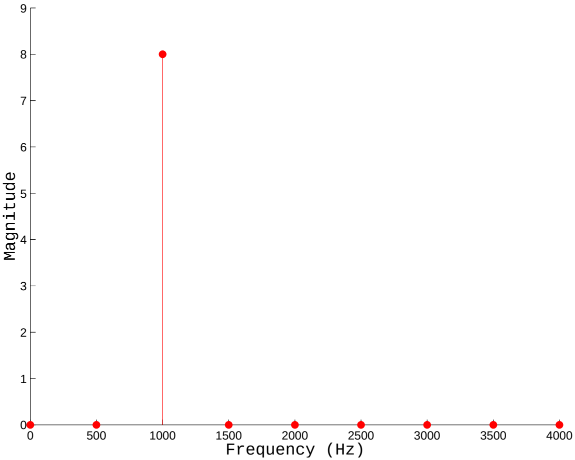
The Discrete Fourier Transform

- (non-periodic) $x[n] \xrightarrow{\text{DFT}} X[k] \xrightarrow{\text{IDFT}} \tilde{x}[n]$ (periodic)
- $X[k]$ and the DTFS of the periodic signal whose fundamental period is $x[n]$ are related by $X[k] = N\, a_k$
- The FFT algorithm is used for computing the DFT coefficients
  - FFT is just an *algorithm*. Wrong to call the result of the FFT as "FFT coefficients" or "FFT spectrum"
  - Wrong usage is well-entrenched in the literature
- We can zero-pad an N-point sequence with $L - N$ zeros and computed the L-point DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi nk/L}$$

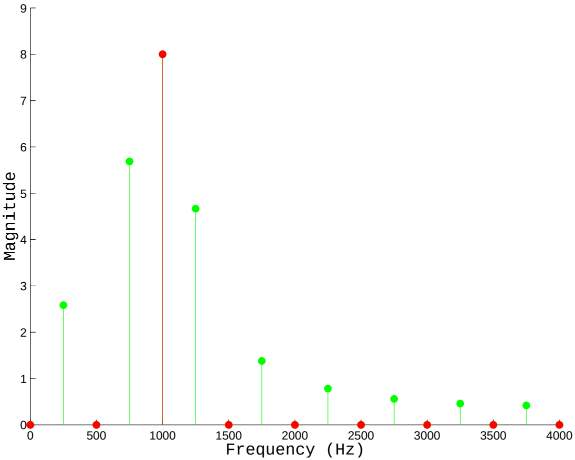
for $k = 0, 1, \ldots, L-1$

Example



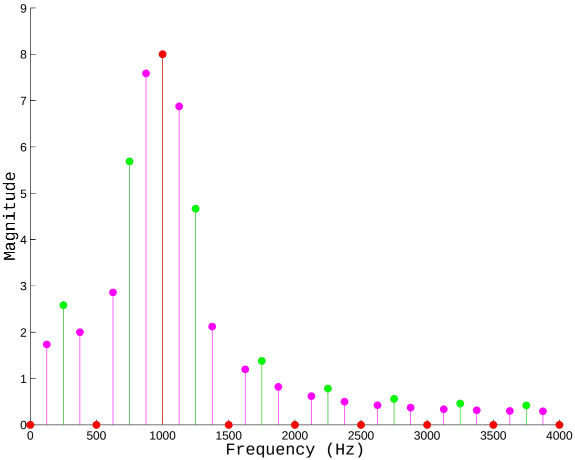
16-pt DFT of $x[n] = \sin(2\pi n/8),\ n = 0, 1, \ldots, 15$

Example



32-pt DFT of $x[n] = \sin(2\pi n/8),\ n = 0, 1, \ldots, 15$

Example

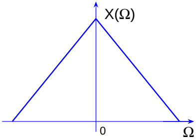


64-pt DFT of $x[n] = \sin(2\pi n/8),\ n = 0, 1, \ldots, 15$

Continuous-Time Fourier Transform

- The "+" and "−" signs in the forward and inverse transform definitions can be switched without changing anything fundamental
- $X(\Omega)$ and $X(j\Omega)$ are commonly used notations to denote the CTFT of $x(t)$
  - If you are given $X(\Omega)$ it is wrong to replace Ω by $j\Omega$ to get $X(j\Omega)$
  - $X(j\Omega)$ notation is useful only to show that it can be obtained from $X(s)$ (Laplace transform) by replacing s by $j\Omega$
- The importance of log scale for the y-axis should be emphasized when plotting magnitude frequency response

- Does $x(t)$ contain DC component?
  - Note that $X(0) \neq 0$
- $x(t)$ does not contain a DC component!
  - If it did, there would be an impulse at $\Omega = 0$



- DC component is defined by

$$\begin{aligned} \text{DC component } &= \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} x(t)\, dt \\ &= \lim_{T \to \infty} \frac{1}{T} X(0) \\ &= 0 \end{aligned}$$

if $X(0)$ is finite

$$\int_{-\infty}^{\infty} \frac{x}{x^2 + a^2} dx \stackrel{?}{=} 0$$

- "The integrand is an odd function and hence the integral is zero"
- Wrong! The above is true only if the limits are finite
- What is zero is the Cauchy Principal Value:

$$\lim_{T \to \infty} \int_{-T}^{T} \frac{x}{x^2 + a^2}\, dx = 0$$

Upper and lower limits approach infinity at the same rate

  - Weaker condition

Relationship Between CTFT and CTFS

- Consider a periodic signal $x(t)$ with FS coefficients a_k
- The CTFT of $x(t)$ is related to the FS coefficients:
  - $X(k\,\Omega_0) = 2\pi \cdot a_k \cdot \delta(\Omega - k\Omega_0)$
  - $X(\Omega) = 0$ for $\Omega \neq k\,\Omega_0$
- Plot of a_k vs. Ω is a simple stem plot
- Plot of $X(\Omega)$ vs. Ω contains impulses, whose strengths are as given above

Continuous-Time Fourier Transform

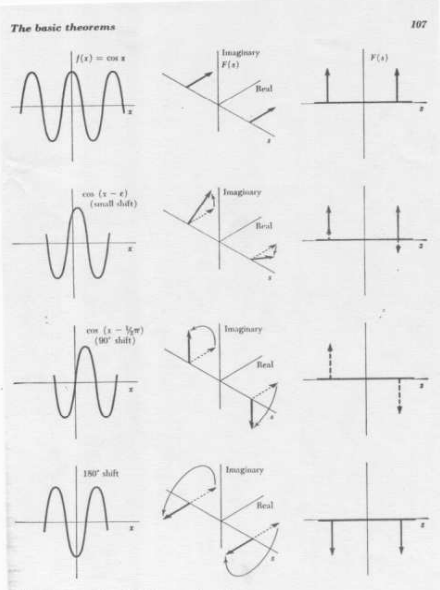


Fig. 6.7 The effect of shifting a cosinusoid.

R. Bracewell, *The Fourier Transform and Its Applications*, McGraw-Hill, 2nd edition, 1986, p. 107

- The DTFT of an aperiodic sequence $x[n]$ is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- $X(\omega + 2\pi) = X(\omega)$ periodicity is 2π

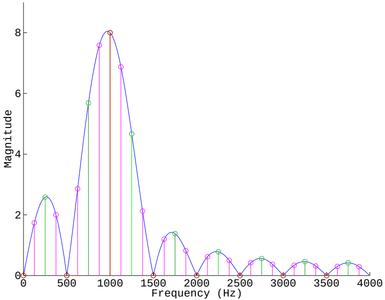
- For a finite duration sequence, the limits go from 0 to $N - 1$

- Notation for DTFT: $X(\omega)$ or $X(e^{j\omega})$
  - If you are given $X(\omega)$, it is wrong to replace ω by $e^{j\omega}$ to get $X(e^{j\omega})$
  - $X(e^{j\omega})$ is useful in relating the DTFT to $X(z)$

- $x[n]$ can be thought of as the FS coefficients of the periodic signal $X(\omega)$

- One can view the DFT coefficients $X[k]$ as samples of the DTFT taken at the points $\omega = 2\pi k/N$:

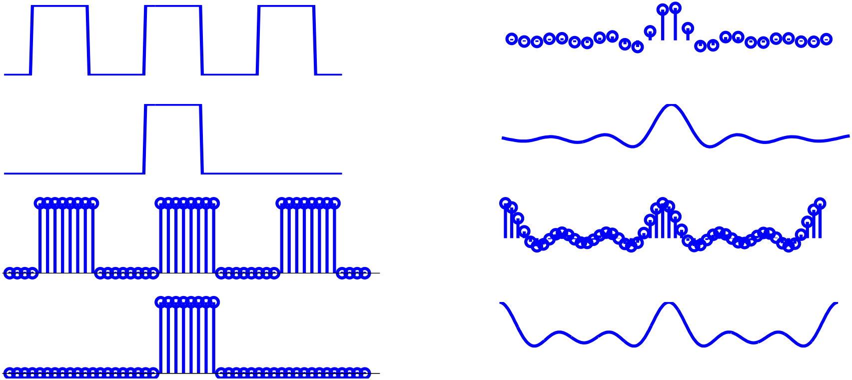
$$
\begin{aligned}
X[k] &= X(\omega)\big|_{\omega=2\pi k/N} \\
&= \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi k/N)n}
\end{aligned}
$$



Sampling Introduces Periodicity in the Time Domain!

- Sampling in the frequency domain leads to periodic repetition in the time domain
- Repetition period is N
- If we sample the DTFT at L ($> N$) points, the repetition period will be L ($> N$)
- If $x[n]$ is of duration N, then $X(\omega)$ has to be sampled at least at N points to avoid aliasing in the time domain
- That is why $X[k] \xrightarrow{\text{IDFT}} \tilde{x}[n]$, and not $x[n]$

Signals and their Transforms



Periodic in one domain $\Longrightarrow$ discrete in the other domain

Discrete in one domain $\overset{?}{\Longrightarrow}$ periodic in the other domain ?

Think about this!