

Weight Distributions

Note Title

(n, k) code C

$$W_C(x, y) = \sum_{i=0}^n A_i x^{n-i} y^i$$

$$A_i = |\{c \in C : wt(c) = i\}|$$

$$= \sum_{\underline{u} \in C} x^{n-wt(\underline{u})} y^{wt(\underline{u})}$$

$$W_{C^\perp}(x, y) = \frac{1}{|C|} W_C(x+y, x-y)$$

Lemma:

$$\sum_{\underline{u} \in C} (-1)^{\underline{u} \cdot \underline{v}} = \begin{cases} |C|, & \underline{v} \in C^\perp \\ 0, & \text{else} \end{cases}$$

Pf (MacWilliams identity)

$$\sum_{\underline{v} \in C^\perp} x^{n-wt(\underline{v})} y^{wt(\underline{v})} = \frac{1}{|C|} \sum_{\underline{v} \in \{0,1\}^n} x^{n-wt(\underline{v})} y^{wt(\underline{v})} \sum_{\underline{u} \in C} (-1)^{\underline{u} \cdot \underline{v}}$$

$$= \frac{1}{|C|} \sum_{\underline{u} \in C} \underbrace{\sum_{\underline{v} \in \{0,1\}^n} (-1)^{\underline{u} \cdot \underline{v}} x^{n - \text{wt}(\underline{v})} y^{\text{wt}(\underline{v})}}$$

$$\underline{u} = [u_1, u_2, \dots, u_n]$$

$$\underline{v} = [v_1, v_2, \dots, v_n]$$

$$\text{wt}(\underline{v}) = \sum_{i=1}^n v_i \quad (*)$$

$$\sum_{v_1=0}^1 \sum_{v_2=0}^1 \dots \sum_{v_n=0}^1 (-1)^{u_1 v_1 + u_2 v_2 + \dots + u_n v_n} x^{n - \sum_{i=1}^n v_i} y^{\sum_{i=1}^n v_i}$$

$$= \sum_{v_1=0}^1 \sum_{v_2=0}^1 \dots \sum_{v_n=0}^1 \left(\prod_{i=1}^n (-1)^{u_i v_i} x^{1-v_i} y^{v_i} \right)$$

$$a_0, a_1, b_0, b_1$$

$$a_0, a_1, b_0, b_1, c_0, c_1$$

$$(a_0 + a_1)(b_0 + b_1)(c_0 + c_1) =$$

$$a_0 b_0 + a_0 b_1 + a_1 b_0 + a_1 b_1$$

$$= (a_0 + a_1)(b_0 + b_1)$$

$$= \prod_{i=1}^n \left(x + (-1)^{u_i} y \right)$$

$$= (x+y)^{n-wt(\underline{u})} (x-y)^{wt(\underline{u})}$$

QED

Ex: ① $C: (n, 1)$ repetition code

$$W_C(x, y) = x^n + y^n$$

$C^\perp: (n, n-1, 2)$
even wt code

$$W_{C^\perp}(x, y) = \frac{(x+y)^n + (x-y)^n}{2}$$

$$= \sum_{i: \text{even}} \binom{n}{i} x^{n-i} y^i$$

② $C: (6, 3)$ code

$$W_C(x, y) = x^6 + 4x^3y^3 + 3x^2y^4$$

$$W_C(x+y, x-y) = (x+y)^6 + 4(x+y)^3(x-y)^3 + 3(x+y)^2(x-y)^4$$

$$= (x+y)^6 + (x+y)^2(x-y)^3(4x+4y+3x-3y)$$

$$\begin{aligned}
 &= (x+y)^6 + (x+y)^2 (x-y)^3 (7x+y) \\
 &= (x+y)^2 \left[(x+y)^4 + (x-y)^3 (7x+y) \right]
 \end{aligned}$$

Ex: $C: (8, 4, 4)$ RM code $(3, 1)$

$$W_c(x, y) = x^8 + 14x^4y^4 + y^8$$

$$\begin{aligned}
 W_c(x+y, x-y) &= (x+y)^8 + 14(x+y)^4(x-y)^4 + (x-y)^8 \\
 &= (x+y)^8 + 14(x^2-y^2)^4 + (x-y)^8 \\
 &\stackrel{!}{=} (x^8 + 14x^4y^4 + y^8) 16
 \end{aligned}$$