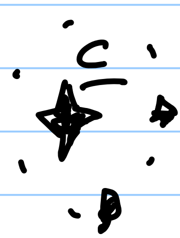
Weight Distributions

C : (n, k) binary code

A_i : # of codewords of C with weight i

$$A_i = \left| \{ \underline{c} \in C : wt(\underline{c}) = i \} \right|$$



e.g. $A_1 = 2$

$A_0 = 1$

$A_i = 0, \ 1 \le i \le d-1$

$$\sum_{i=0}^{n} A_i = 2^k$$

Ex: ① $(n,1)$ repetition code

$$A_0 = 1 \qquad A_n = 1$$

② $(n, n-1)$ even weight code

$$A_{2i} = \binom{n}{2i} \qquad i = 0, 1, \ldots, \lfloor \tfrac{n}{2} \rfloor$$

③
$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad A_0 = 1,\ A_3 = 4 \quad A_4 = 3$$

$C = \{000000, 101001, 011010, 110011, 110100, 011101, 101110, 000111\}$

$C^\perp = \{000000, 100101, 010110, 001011, 110011, 101110, 011101, 111000\}$

$$A'_0 = 1,\ A'_3 = 4,\ A'_4 = 3$$

Notation: $A'_i = |\{\underline{c} \in C^\perp : wt(\underline{c}) = i\}|$.

Weight distribution polynomial

$$W_C(x, y) = \sum_{i=0}^{n} A_i\, x^{n-i} y^i$$

$$W_C(x,y) = \sum_{\underline{c} \in C} x^{n-wt(\underline{c})} y^{wt(\underline{c})}$$

Ex: (1) C: $(n,1)$ repetition code

$$W_C(x,y) = x^n + y^n$$

(2) $C^\perp$: $(n, n-1)$ even-wt code

$$W_{C^\perp}(x,y) = \sum_{i=0,1,2,\dots,\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} x^{n-2i} y^{2i}$$

$$= \sum_{i:\text{ even}} \binom{n}{i} x^{n-i} y^{i}$$

$$= \frac{(x+y)^n + (x-y)^n}{2}$$

(3) $(7,4,3)$ Hamming Code

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$A_0 = 1$, $A_3 = 7$, $A_4 = 7$

$A_7 = 1$

$C : (7,4,3)$ Hamming Code

$$W_C(x,y) = x^7 + 7x^4y^3 + 7x^3y^4 + y^7$$

$C^\perp : (7,3,4)$

Put $x = x+y$

$y = x-y$

- divide by 16

$$W_{C^\perp}(x,y) = x^7 + 7x^3y^4$$

<u>MacWilliams identity</u>

$$W_{C^\perp}(x,y) = \frac{1}{|C|} W_C(x+y, x-y)$$

$$\rightarrow \sum A_i' x^{n-i} y^i = \frac{1}{|C|} \sum A_i (x+y)^{n-i} (x-y)^i$$

$$\boxed{\sum_{\underline{u} \in C^\perp} x^{n-wt(\underline{u})} y^{wt(\underline{u})} = \frac{1}{|C|} \sum_{\underline{u} \in C} (x+y)^{n-wt(\underline{u})} (x-y)^{wt(\underline{u})}}$$

Lemma: $\underline{u}, \underline{v} \in \{0,1\}^n$

$$\sum_{\underline{u} \in C} (-1)^{\underline{u} \cdot \underline{v}} = \begin{cases} |C|, & \underline{v} \in C^{\perp} \\ 0, & \underline{v} \notin C^{\perp} \end{cases}$$

Pf: Set: $\{\underline{u} \in C : \underline{u} \cdot \underline{v} = 0\}$ is a subcode with one other coset, if $\underline{v} \notin C^{\perp}$

Pf (MacWilliams' identity)

$$\sum_{\underline{u} \in C^{\perp}} x^{n-wt(\underline{u})} y^{wt(\underline{u})} = \frac{1}{|C|} \sum_{\underline{v} \in \{0,1\}^n} x^{n-wt(\underline{v})} y^{wt(\underline{v})} \cdot \sum_{\underline{u} \in C} (-1)^{\underline{u} \cdot \underline{v}}$$

$$= \frac{1}{|C|} \sum_{\underline{u} \in C} \underbrace{\sum_{\underline{v} \in \{0,1\}^n} (-1)^{\underline{u} \cdot \underline{v}} x^{n-wt(\underline{v})} y^{wt(\underline{v})}}$$