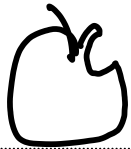
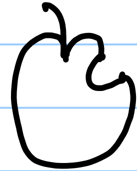
Reed-Muller Codes

Note Title



$\underline{RM(r, m)}$



$$c_i = f(v_1, v_2, \ldots, v_m), \quad \deg f \leq r$$

$$\underline{c} = [\, c_0 \quad c_1 \quad c_2 \quad \cdots \quad c_{2^m - 1} \,]$$

$$\begin{array}{cccc}
v_1 & 0 & \cdots & 1 \\
v_2 & 0 & \cdots & 1 \\
\vdots & \vdots & & \vdots \\
v_m & 0 & \cdots & 1
\end{array}$$

Majority logic decoding

(n, k) code C ; $(n, n-k)$ $C^{\perp}$

$\underline{c} \in C$ $\qquad$ $\underline{v} \in C^{\perp}$

$$\underline{c} \cdot \underline{v} = 0$$

con

Fixed $\underline{v}$.

Equation $\longrightarrow$ $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$

Satisfied by all $\underline{c} \in C$

$$\sum_{i : v_i = 1} c_i = 0$$

parity-check

Parity-checks "orthogonal" on the i-th bit c_i

$$\underline{v}_1 + \underline{v}_2 \in C^{\perp} \setminus \{\underline{0}\}$$

s.t. $v_{1,i} = v_{2,i} = 1$

and $\{j : v_{1,j} = 1\} \cap \{j' : v_{2,j'} = 1\} = \{i\}$

$$\begin{bmatrix} v_{11} v_{21} & v_{12} v_{22} & \cdots & v_{1n} v_{2n} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\uparrow$$

i-th place

Suppose there are J parity checks orthogonal on bit i.

$1: \quad c_i + c_{j_1} + c_{j_2} + \cdots + c_{j_{n_1}} = 0$

no intersection

$2: \quad c_i + c_{j_{n_1+1}} + c_{j_{n_1+2}} + \cdots + c_{j_{n_1+n_2}} = 0$

$\vdots \qquad \vdots$

$J: \quad c_i + c_? + c_? + \cdots + c_? = 0$

Decoder: $\underline{r} = \underline{c} + \underline{e}$ $\quad 1: \hat{c}_i^{(1)} = r_{j_1} + r_{j_2} + \cdots + r_{j_{n_1}}$

$$2: \hat{c}_i^{(2)} = r_{j_{n_1+1}} + r_{j_{n_1+2}} + \cdots + r_{j_{n_1+n_2}}$$

$$\vdots \qquad\qquad \vdots$$

$$J: \hat{c}_i^{(J)} = r_{?} + r_{?} + \cdots + r_{?}$$

Majority logic: $\hat{c}_i = \text{maj}\left\{\hat{c}_i^{(1)}, \hat{c}_i^{(2)}, \ldots, \hat{c}_i^{(J)}\right\}$

Claim: if $wt(\underline{e}) \leq \left\lfloor \frac{J}{2} \right\rfloor$, then $\hat{c}_i = c_i$

→ If code is cyclic, simply shift the checks for i.