Lecture 2

Note Title 7/30/2008

$$\underset{[b_1\, b_2 \ldots]}{\underline{b}} \longrightarrow \boxed{Tx} \xrightarrow{x(t)} \boxed{\text{Channel}} \xrightarrow{y(t)} \boxed{Rx} \longrightarrow \underset{[\hat{b}_1\, \hat{b}_2 \ldots]}{\hat{\underline{b}}}$$

$x(t) = Tx(\underline{b})$ $\quad\quad$ $\hat{\underline{b}} = Rx(y(t))$

$\left.\begin{array}{l}\text{(Avg) Power of } x(t) = P_x \\ \text{BW of } x(t) = W\end{array}\right\}$ constrained

$$\boxed{\begin{array}{c} y(t) = x(t) * h(t) \\ + n(t) \end{array}}$$

$$\downarrow$$

$$\boxed{y(t) = x(t) + n(t)}$$

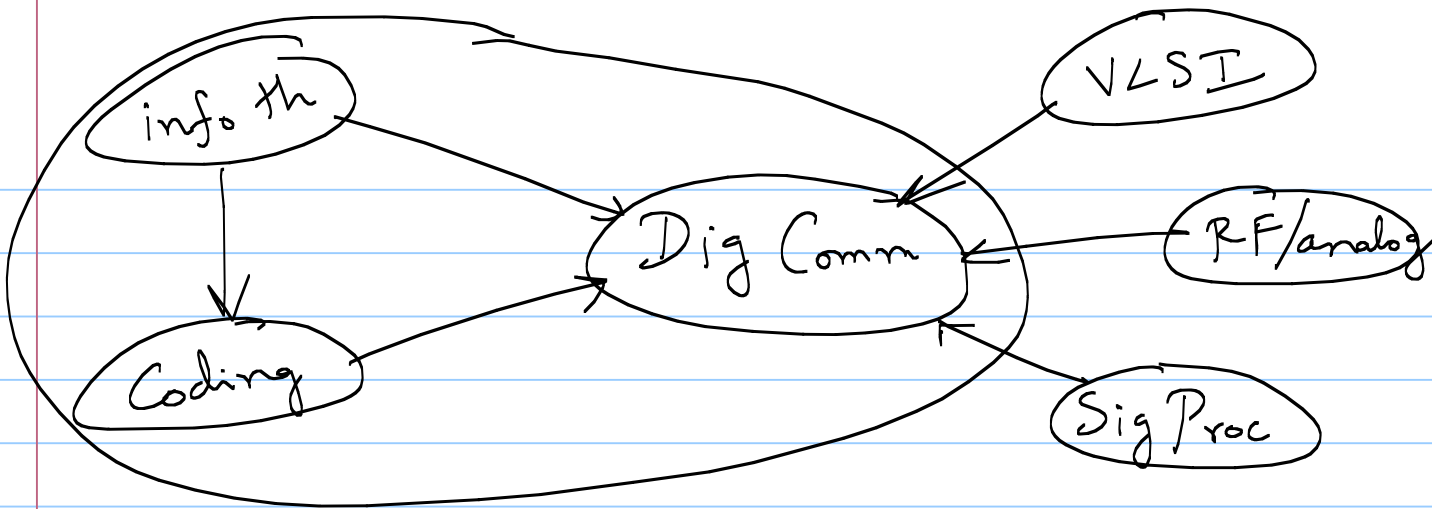
Bit Rate $= R$

Probability of error $= Pr(\hat{b}_i \neq b_i)$

Noise Power $= P_n$

$$R \leq W \log_2\left(1 + \frac{P_x}{P_n}\right)$$

bps



Preliminaries:

1) Complex vector space $\mathbb{C}^n$

$$\underline{z} = [\, z[1] \;\; z[2] \;\; \cdots \;\; z[n] \,]^T \qquad z[i] \in \mathbb{C}$$

$$z[i] = x[i] + j\,y[i] = r[i]\, e^{j\theta[i]}$$

$$\langle \underline{z}_1, \underline{z}_2 \rangle = \sum_{i=1}^{n} z_1[i]\, z_2^*[i] = \underline{z}_2^H \underline{z}_1$$

$$\|\underline{z}\| = \langle \underline{z}, \underline{z} \rangle$$

2) Function space $\mathbb{L}_2$

$$\mathbb{L}_2 = \left\{ x(t) : \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty \right\}$$

Complex-valued

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) \, y^*(t) \, dt$$

$$\| x(t) \|_2 = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Function space $\mathbb{L}_\infty = \{ x(t) : |x(t)| < \infty \}$

contains $e^{j\omega t}$

Cauchy-Schwartz: vectors $s, r \in$ inner product space

$$|\langle s, r \rangle| \leq \|s\| \, \|r\|$$

Convolution: $s(t), r(t)$

$$q(t) = (s * r)(t) = \int_{-\infty}^{\infty} s(u)\, r(t-u)\, du$$

$$\big(= s(t) * r(t) \big)$$

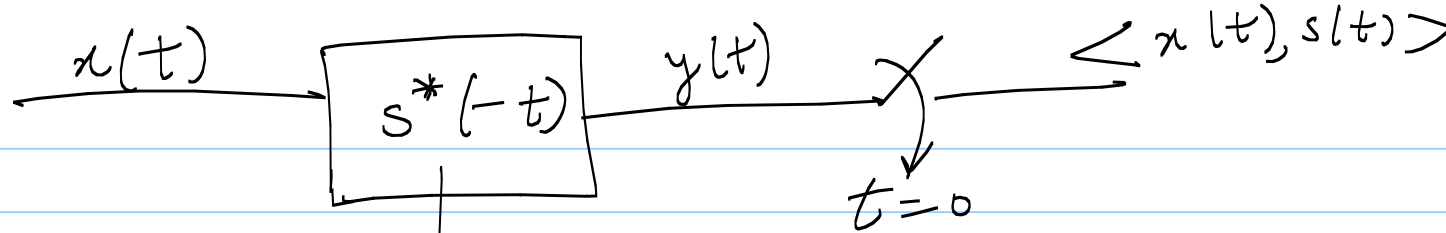
Delta Function: $\displaystyle\int_{-\infty}^{\infty} \delta(t - t_0)\, s(t)\, dt = s(t_0)$

Linear filter: $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$

impulse response

Computing inner product by filtering + sampling:

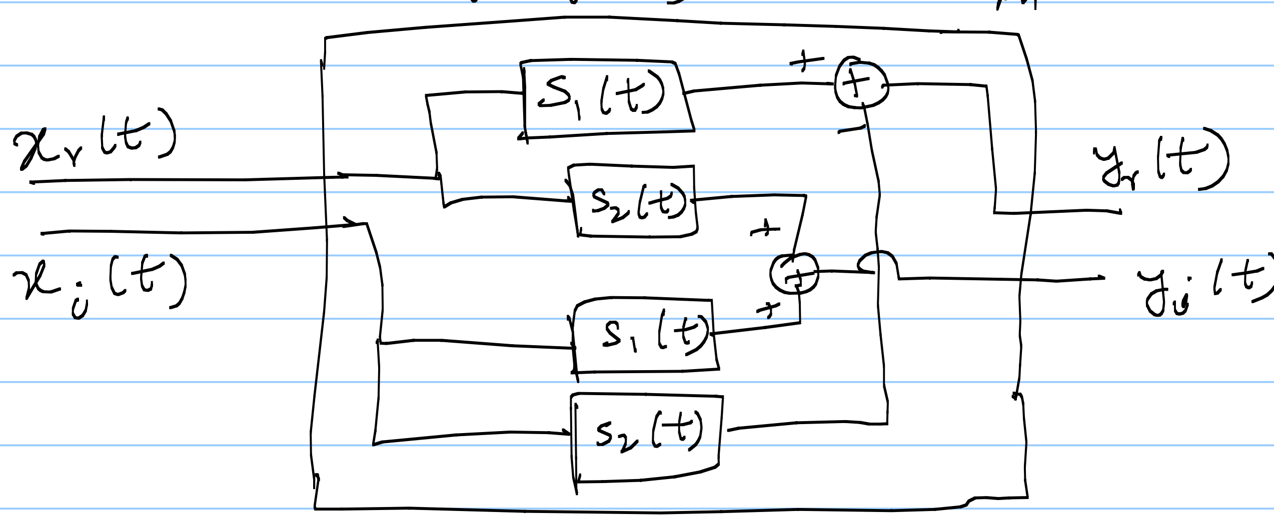
$$\langle x(t), s(t) \rangle = \int_{-\infty}^{\infty} x(u)\, s^*(u)\, du$$



Matched filter (matched to s(t))

$$s_{MF}(t) = s^*(-t)$$

$$x(t) = x_r(t) + j\,x_i(t) \qquad s_{MF}(t) = s_1(t) + j\,s_2(t)$$



Sinc:

$$\mathrm{sinc}(t) = \begin{cases} \frac{\sin \pi t}{\pi t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

Rect:

$$\underset{[a,b]}{\mathrm{rect}}(t) = \begin{cases} 1 & a \leq t \leq b \\ 0 & \text{else} \end{cases}$$

Unit step:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{else} \end{cases}$$

Fourier Transform:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, df$$

$x(t) \in \mathbb{L}_2$
$X(f) \in \mathbb{L}_2$

$$\langle x_1(t), x_2(t) \rangle = \langle X_1(f), X_2(f) \rangle$$

Properties of FT: 1) $\text{rect}_{[-\frac{T}{2},\frac{T}{2}]}(t) \xleftrightarrow{FT} T\,\text{sinc}(fT)$

$$\delta(t) \longleftrightarrow 1$$

2) $x(t) \longleftrightarrow X(f)$

$$X(-t) \longleftrightarrow x(f)$$

$$x^*(t) \longleftrightarrow X^*(-f)$$

$$x^*(-t) \longleftrightarrow X^*(f)$$

$x(t)$: real valued

$$X(f) = X^*(-f)$$

$$\text{Re}\{X(f)\} = \text{Re}\{X(-f)\}$$

$$\text{Im}\{X(f)\} = -\text{Im}\{X(f)\}$$

3) $x_1(t) * x_2(t) \longleftrightarrow X_1(f)\,X_2(f)$

$$x_1(t)\,x_2(t) \longleftrightarrow X_1(f) * X_2(f)$$

$$x(t-t_0) \longleftrightarrow e^{-j2\pi f t_0} X(f)$$

$$x(t)\, e^{j2\pi f_0 t} \longleftrightarrow X(f-f_0)$$

(3) Autocorrelation and Energy spectrum

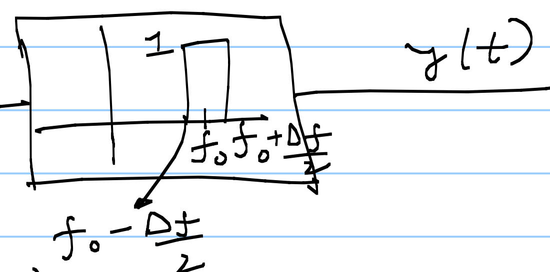
$x(t)$

$$R_x(\tau) = \int_{-\infty}^{\infty} x(u)\, x^*(u-\tau)\, du$$

$$= x(\tau) * x^*(-\tau)$$

$$FT\{R_x(\tau)\} = |X(f)|^2$$

Energy spectrum:



$$\text{Energy spectrum of } x(t) \text{ at } f_0 = \frac{\|y(t)\|_2}{\Delta f}$$

$$= |X(f_0)|^2$$

Baseband and Passband signals

$x(t)$: baseband $\quad |X(f)| = 0, \ |f| > W$ for some $W > 0$

LTI filter $h(t)$: baseband if $h(t)$: baseband.

$x(t)$: passband $\quad |X(f)| = 0, \ |f \pm f_c| > W$ for $f_c > W > 0$

LTI filter $h(t)$: passband if $h(t)$: passband

→ real baseband and passband signals

complex baseband