Solutions to Problem Set 6

EE419: Digital Communication Systems

Check the solutions for possible bugs!

1. **ZF-LE:**

   (a) Filter: $\dfrac{1}{1-cz^{-1}}$.

   MSE=$< \dfrac{S_n}{(1-cz^{-1})(1-c^*z)} >_A$. Doing partial fraction expansion,

$$\frac{N_0}{(1-cz^{-1})(1-c^*z)} = \frac{N_0 z}{(z-c)(1-c^*z)} = \frac{\frac{N_0 c}{1-|c|^2}}{z-c} + \frac{\frac{N_0}{1-|c|^2}}{1-c^*z}.$$

   Rewriting in a different form, we get

$$\frac{N_0}{(1-cz^{-1})(1-c^*z)} = \frac{N_0}{1-|c|^2}\left(\frac{cz^{-1}}{1-cz^{-1}} + \frac{1}{1-c^*z}\right).$$

   For $|cz^{-1}| < 1$ or $|z| > |c|$, we have

$$\frac{1}{1-cz^{-1}} = 1 + cz^{-1} + c^2 z^{-2} + \cdots .$$

   For $|c^*z| < 1$ or $|z| < 1/|c|$, we get

$$\frac{1}{1-c^*z} = 1 + c^*z + (c^*)^2 z^2 + \cdots .$$

   The coefficient of z^0 in the stable expansion of $\dfrac{S_n}{(1-cz^{-1})(1-c^*z)}$ for $|c| < |z| < 1/|c|$ is the MSE given as follows:

$$\text{MSE} = \frac{N_0}{1-|c|^2}.$$

   (b) Filter: $\dfrac{1}{1-cz^{-1}}$.

   MSE=$< \dfrac{S_n}{(1-cz^{-1})(1-c^*z)} >_A = N_0$.

   (f) Filter: $1-cz^{-1}$.

   MSE=$< N_0(1-cz^{-1})(1-c^*z) >_A = N_0(1+|c|^2)$.

   ZF-DFE:

   (a) $|c| < 1$: Precursor = 1; Postcursor = $-cz^{-1}$; MSE = $< \dfrac{N_0}{(1-cz^{-1})(1-c^*z)} >_G = N_0$.

   $|c| > 1$: Write $H(z) = 1-cz^{-1} = -cz^{-1}(1-z/c)$. Hence, $H_0 = -c$, $H_{\min}(z) = 1$, $H_{\max}(z) = 1-z/c$ and $H^*_{\max}(1/z^*) = 1-z^{-1}/c^*$. Precursor = $\dfrac{1-z^{-1}/c^*}{-c(1-z/c)}$; Postcursor = $-z^{-1}/c^*$;

$$MSE = \frac{N_0}{(1-cz^{-1})(1-c^*z)} = \frac{N_0}{|c|^2(1-z^{-1}/c^*)(1-z/c)} = N_0/|c|^2.$$

(b) $|c| < 1$: $S_n(z) = N_0(1 - cz^{-1})(1 - c^* z)$ is a valid spectral factorization. Hence, $\gamma_n^2 = N_0$ and $M_n(z) = 1 - cz^{-1}$. Precursor $= \frac{1}{1 - cz^{-1}}$; Postcursor $= 0$; MSE $= N_0$.

$|c| > 1$: $S_n(z) = N_0|c|^2(1 - z^{-1}/c^*)(1 - z/c)$ is a valid spectral factorization. Hence, $\gamma_n^2 = N_0|c|^2$ and $M_n(z) = 1 - z^{-1}/c^*$. Similarly, $H_0 = -c$, $H_{\min}(z) = 1$, $H_{\max}(z) = 1 - z/c$ and $H^*_{\max}(1/z^*) = 1 - z^{-1}/c^*$.

$$\text{Precursor} = \frac{1 - z^{-1}/c^*}{-c(1 - z/c)} \frac{1}{(1 - z^{-1}/c^*)} = \frac{z^{-1}}{1 - cz^{-1}};$$

$$\text{Postcursor} = \frac{1 - z^{-1}/c^*}{1 - z^{-1}/c^*} - 1 = 0;$$

$$\text{MSE} = < \frac{N_0(1 - cz^{-1})(1 - c^* z)}{(1 - cz^{-1})(1 - c^* z)} >_G = N_0.$$

(f) $|c| < 1$: Precursor $= 1$; Postcursor $= \frac{1}{1 - cz^{-1}} - 1$; MSE$=< N_0(1 - cz^{-1})(1 - c^* z) >_G = N_0$.

MMSE-LE:

(a) $S_z(z) = (1 - cz^{-1})(1 - c^* z) + N_0 = \gamma_z^2(1 - \alpha z^{-1})(1 - \alpha^* z)$ (Exercise: Find α and γ_z^2 in terms of c and N_0).

$$\text{Filter} = \frac{1 - c^* z}{(1 - cz^{-1})(1 - c^* z) + N_0};$$

MSE $= < \frac{N_0}{(1 - cz^{-1})(1 - c^* z) + N_0} >_A$. Simplifying,

$$\text{MSE} = < \frac{N_0}{\gamma_z^2(1 - \alpha z^{-1})(1 - \alpha^* z)} >_A = \frac{N_0}{\gamma_z^2(1 - |\alpha|^2)}.$$

(b) $S_z(z) = (1 - cz^{-1})(1 - c^* z) + N_0(1 - cz^{-1})(1 - c^* z) = (N_0 + 1)(1 - cz^{-1})(1 - c^* z)$.

$$\text{Filter} = \frac{1}{(N_0 + 1)(1 - cz^{-1})};$$

$$\text{MSE} = < \frac{N_0(1 - cz^{-1})(1 - c^* z)}{(N_0 + 1)(1 - cz^{-1})(1 - c^* z)} >_A = \frac{N_0}{N_0 + 1}.$$

(f) $S_z(z) = \frac{1}{(1 - cz^{-1})(1 - c^* z)} + N_0 = \gamma_z^2 \left(\frac{1 - \alpha z^{-1}}{1 - cz^{-1}}\right)\left(\frac{1 - \alpha^* z}{1 - c^* z}\right)$ (Exercise: Find α and γ_z^2 in terms of c and N_0).

$$\text{Filter} = \frac{1}{1 - c^* z} \frac{(1 - cz^{-1})(1 - c^* z)}{1 + N_0(1 - cz^{-1})(1 - c^* z)} = \frac{(1 - cz^{-1})}{1 + N_0(1 - cz^{-1})(1 - c^* z)};$$

$$\text{MSE} = < \frac{N_0(1 - cz^{-1})(1 - c^* z)}{\gamma_z^2(1 - \alpha z^{-1})(1 - \alpha^* z)} >_A .$$

Using partial fraction expansions, we can write

$$\frac{N_0(1 - cz^{-1})(1 - c^* z)}{\gamma_z^2(1 - \alpha z^{-1})(1 - \alpha^* z)} = \frac{N_0}{\gamma_z^2(1 - |\alpha|^2)}(1 + |c|^2 - cz^{-1} - c^* z)(\cdots + \alpha^* z + 1 + \alpha z^{-1} + \cdots).$$

From the above, we identify the coefficient of z^0 as MSE$= \frac{N_0(1 + |c|^2 - c\alpha^* - c^* \alpha)}{\gamma_z^2(1 - |\alpha|^2)}$.

MMSE-DFE:

(a) $S_z(z) = (1 - cz^{-1})(1 - c^*z) + N_0 = \gamma_z^2(1 - \alpha z^{-1})(1 - \alpha^* z)$ and $M_n(z) = 1$, $\gamma_n^2 = 1$.

$$\text{Precursor} = \frac{1 - c^*z}{\gamma_z^2(1 - \alpha^* z)};$$

Postcursor $= -\alpha z^{-1}$ and $\text{MSE} = < \dfrac{N_0}{\gamma_z^2(1 - \alpha z^{-1})(1 - \alpha^* z)} >_G = N_0/\gamma_z^2$.

(b) $|c| < 1$: $S_z(z) = (1 - cz^{-1})(1 - c^*z) + N_0(1 - cz^{-1})(1 - c^*z) = (N_0 + 1)(1 - cz^{-1})(1 - c^*z)$ and $S_n(z) = N_0(1 - cz^{-1})(1 - c^*z)$ are valid spectral factorizations. Hence,

$$\text{Precursor} = \frac{1 - c^*z}{(N_0 + 1)(1 - c^*z)(1 - cz^{-1})} = \frac{1}{(N_0 + 1)(1 - cz^{-1})};$$

Postcursor=0;

$$\text{MSE} = < \frac{N_0(1 - cz^{-1})(1 - c^*z)}{(N_0 + 1)(1 - cz^{-1})(1 - c^*z)} >_G = \frac{N_0}{N_0 + 1}.$$

$|c| > 1$: $S_z(z) = (N_0 + 1)|c|^2(1 - z^{-1}/c^*)(1 - z/c)$ and $S_n(z) = N_0|c|^2(1 - z^{-1}/c^*)(1 - z^{-1}/c)$ are valid spectral factorizations. Hence,

$$\text{Precursor} = \frac{1 - c^*z}{(N_0 + 1)|c|^2(1 - z/c)(1 - z^{-1}/c^*)} = \frac{z^{-1}}{(N_0 + 1)(1 - cz^{-1})};$$

Postcursor=0;

$$\text{MSE} = < \frac{N_0(1 - cz^{-1})(1 - c^*z)}{(N_0 + 1)(1 - cz^{-1})(1 - c^*z)} >_G = \frac{N_0}{N_0 + 1}.$$

(f) $S_z(z) = \dfrac{1}{(1 - cz^{-1})(1 - c^*z)} + N_0 = \gamma_z^2 \left(\dfrac{1 - \alpha z^{-1}}{1 - \beta z^{-1}}\right)\left(\dfrac{1 - \alpha^* z}{1 - \beta^* z}\right)$ (Exercise: Find α, β and γ_z^2 in terms of c and N_0. Distinguish between $|c| < 1$ and —c—¿1).

$$\text{Precursor} = \frac{1 - \beta^* z}{\gamma_z^2(1 - c^*z)(1 - \alpha^* z)};$$

$$\text{Postcursor} = \frac{1 - \alpha z^{-1}}{1 - \beta z^{-1}} - 1;$$

$$\text{MSE} = < \frac{N_0(1 - \beta z^{-1})(1 - \beta^* z)}{\gamma_z^2(1 - \alpha z^{-1})(1 - \alpha^* z)} >_G = \frac{N_0}{\gamma_z^2}.$$

2-4. No solution provided.