Solutions to Problem Set 5

EE419: Digital Communication Systems

Check the solutions for possible bugs!

1. No solution provided.

2. No solution provided.

3. (a) Assuming symbols are equally likely *a priori*, the probability of symbol error is given by

$$P_e = \frac{6}{8}(2Q(1/\sigma)) + \frac{2}{8}(Q(1/\sigma)) = \frac{7}{4}Q(1/\sigma).$$

   (b) One possible Gray labeling is $b_0b_1b_2$ =000,001,010,011,111,110,100,101 from left to right. The LLR for b_0 (assuming bits are *iid* uniform *a priori*) is given by

$$\begin{aligned}\mathrm{LLR}(b_0) &= \log\frac{f(r|b_0=0)}{f(r|b_0=1)}\\ &= \log\frac{e^{-(r+7)^2/2\sigma^2}+e^{-(r+5)^2/2\sigma^2}+e^{-(r+3)^2/2\sigma^2}+e^{-(r+1)^2/2\sigma^2}}{e^{-(r-7)^2/2\sigma^2}+e^{-(r-5)^2/2\sigma^2}+e^{-(r-3)^2/2\sigma^2}+e^{-(r-1)^2/2\sigma^2}}.\end{aligned}$$

   For b_1,

$$\begin{aligned}\mathrm{LLR}(b_1) &= \log\frac{f(r|b_1=0)}{f(r|b_1=1)}\\ &= \log\frac{e^{-(r+7)^2/2\sigma^2}+e^{-(r+5)^2/2\sigma^2}+e^{-(r-5)^2/2\sigma^2}+e^{-(r-7)^2/2\sigma^2}}{e^{-(r+3)^2/2\sigma^2}+e^{-(r+1)^2/2\sigma^2}+e^{-(r-1)^2/2\sigma^2}+e^{-(r-3)^2/2\sigma^2}}.\end{aligned}$$

   For b_2,

$$\begin{aligned}\mathrm{LLR}(b_2) &= \log\frac{f(r|b_2=0)}{f(r|b_2=1)}\\ &= \log\frac{e^{-(r+7)^2/2\sigma^2}+e^{-(r+3)^2/2\sigma^2}+e^{-(r-3)^2/2\sigma^2}+e^{-(r-5)^2/2\sigma^2}}{e^{-(r+5)^2/2\sigma^2}+e^{-(r+1)^2/2\sigma^2}+e^{-(r-1)^2/2\sigma^2}+e^{-(r-7)^2/2\sigma^2}}.\end{aligned}$$

   Probability of error for b_0 is given by

$$\Pr\{\text{Error in } b_0\} = \frac{1}{4}(Q(1/\sigma)+Q(3/\sigma)+Q(5/\sigma)+Q(7/\sigma)).$$

   Probability of error for b_1 is given by

$$\begin{aligned}\Pr\{\text{Error in } b_1\} &= \frac{1}{4}Q(1/\sigma)+\frac{1}{4}Q(3/\sigma)+\frac{1}{4}(Q(1/\sigma)+Q(7/\sigma))+\frac{1}{4}(Q(3/\sigma)+Q(5/\sigma))\\ &= 0.5(Q(1/\sigma)+Q(3/\sigma))+0.25(Q(5/\sigma)+Q(7/\sigma)).\end{aligned}$$

Probability of error for b_2 is given by

$$\frac{1}{8}(Q(1/\sigma) - Q(3/\sigma) + Q(5/\sigma) - Q(9/\sigma) + Q(13/\sigma)) + \frac{1}{8}(Q(1/\sigma) + Q(1/\sigma) - Q(3/\sigma) + Q(7/\sigma) - Q(11/\sigma)) +$$
$$\frac{1}{8}(Q(1/\sigma) - Q(3/\sigma) + Q(1/\sigma) - Q(5/\sigma) + Q(9/\sigma)) + \frac{1}{8}(Q(3/\sigma) - Q(5/\sigma) + Q(3/\sigma) - Q(7/\sigma)) +$$
$$\frac{1}{8}(Q(1/\sigma) - Q(5/\sigma) + Q(3/\sigma) - Q(5/\sigma) + Q(7/\sigma)) + \frac{1}{8}(Q(3/\sigma) + Q(1/\sigma) - Q(5/\sigma) + Q(7/\sigma) - Q(9/\sigma)) +$$
$$\frac{1}{8}(Q(1/\sigma) + Q(3/\sigma) - Q(7/\sigma) + Q(9/\sigma) - Q(11/\sigma)) + \frac{1}{8}(Q(1/\sigma) - Q(5/\sigma) + Q(9/\sigma) - Q(11/\sigma) + Q(13/\sigma))$$
$$= \frac{9}{8}Q(1/\sigma) + \frac{1}{4}Q(3/\sigma) - \frac{5}{8}Q(5/\sigma) + \frac{1}{8}Q(7/\sigma) + \frac{1}{8}Q(9/\sigma) - \frac{3}{8}Q(11/\sigma) + \frac{1}{4}Q(13/\sigma).$$

(c) Binary labeling $b_0b_1b_2$ =000,001,010,011,100,101,110,111 from left to right. The LLR for b_0 (assuming bits are *iid* uniform *a priori*) is given by

$$\begin{aligned}\mathrm{LLR}(b_0) &= \log \frac{f(r|b_0=0)}{f(r|b_0=1)} \\ &= \log \frac{e^{-(r+7)^2/2\sigma^2} + e^{-(r+5)^2/2\sigma^2} + e^{-(r+3)^2/2\sigma^2} + e^{-(r+1)^2/2\sigma^2}}{e^{-(r-7)^2/2\sigma^2} + e^{-(r-5)^2/2\sigma^2} + e^{-(r-3)^2/2\sigma^2} + e^{-(r-1)^2/2\sigma^2}}.\end{aligned}$$

For b_1,

$$\begin{aligned}\mathrm{LLR}(b_1) &= \log \frac{f(r|b_1=0)}{f(r|b_1=1)} \\ &= \log \frac{e^{-(r+7)^2/2\sigma^2} + e^{-(r+5)^2/2\sigma^2} + e^{-(r-1)^2/2\sigma^2} + e^{-(r-3)^2/2\sigma^2}}{e^{-(r+3)^2/2\sigma^2} + e^{-(r+1)^2/2\sigma^2} + e^{-(r-5)^2/2\sigma^2} + e^{-(r-7)^2/2\sigma^2}}.\end{aligned}$$

For b_2,

$$\begin{aligned}\mathrm{LLR}(b_2) &= \log \frac{f(r|b_2=0)}{f(r|b_2=1)} \\ &= \log \frac{e^{-(r+7)^2/2\sigma^2} + e^{-(r+3)^2/2\sigma^2} + e^{-(r-1)^2/2\sigma^2} + e^{-(r-5)^2/2\sigma^2}}{e^{-(r+5)^2/2\sigma^2} + e^{-(r+1)^2/2\sigma^2} + e^{-(r-3)^2/2\sigma^2} + e^{-(r-7)^2/2\sigma^2}}.\end{aligned}$$

Probability of error for b_0 is given by

$$\Pr\{\text{Error in } b_0\} = \frac{1}{4}(Q(1/\sigma) + Q(3/\sigma) + Q(5/\sigma) + Q(7/\sigma)).$$

Probability of error for b_1 is given by

$$\begin{aligned}\Pr\{\text{Error in } b_1\} &= \frac{1}{4}(Q(3/\sigma) - Q(7/\sigma) + Q(11/\sigma)) + \frac{1}{4}(Q(1/\sigma) - Q(5/\sigma) + Q(9/\sigma)) + \\ &\quad \frac{1}{4}(Q(1/\sigma) + Q(3/\sigma) - Q(7/\sigma)) + \frac{1}{4}(Q(3/\sigma) + Q(1/\sigma) - Q(5/\sigma)) \\ &= \frac{3}{4}(Q(1/\sigma) + Q(3/\sigma)) - \frac{1}{2}(Q(5/\sigma) + Q(7/\sigma)) + \frac{1}{4}(Q(9/\sigma) + Q(11/\sigma)).\end{aligned}$$

Probability of error for b_2 is given by

$$\frac{1}{4}(Q(1/\sigma) - Q(3/\sigma) + Q(5/\sigma) - Q(7/\sigma) + Q(9/\sigma) - Q(11/\sigma) + Q(13/\sigma)) +$$
$$\frac{1}{4}(Q(1/\sigma) + Q(1/\sigma) - Q(3/\sigma) + Q(5/\sigma) - Q(7/\sigma) + Q(9/\sigma) - Q(11/\sigma)) +$$
$$\frac{1}{4}(Q(1/\sigma) - Q(3/\sigma) + Q(1/\sigma) - Q(3/\sigma) + Q(5/\sigma) - Q(7/\sigma) + Q(9/\sigma)) +$$
$$\frac{1}{4}(Q(1/\sigma) - Q(3/\sigma) + Q(5/\sigma) - Q(7/\sigma) + Q(1/\sigma) - Q(3/\sigma) + Q(5/\sigma) - Q(7/\sigma))$$
$$= \frac{7}{4}Q(1/\sigma) - \frac{3}{2}Q(3/\sigma) + \frac{5}{4}Q(5/\sigma) - \frac{5}{4}Q(7/\sigma) + \frac{3}{4}Q(9/\sigma) - \frac{1}{2}Q(11/\sigma) + \frac{1}{4}Q(13/\sigma).$$

(d) Compare the $Q(1/\sigma)$ terms in each expression.

4. Similar to the previous problem.

5. (a) When A is a known constant, we have BPSK transmission. This has been discussed extensively in class.

   (b) When $A \in \{\pm 1\}$ is a discrete random variable (independent of X and N) with $p = \Pr\{A = 1\}$, the received signal constellation is 1D with two points $\{-1, 1\}$. To find the decision threshold y, the condtion is that

$$\begin{aligned} f(y|X=-1) &= f(y|X=1) \\ pe^{-(y+1)^2/2\sigma^2} + (1-p)e^{-(y-1)^2/2\sigma^2} &= pe^{-(y-1)^2/2\sigma^2} + (1-p)e^{-(y+1)^2/2\sigma^2} \\ (1-2p)e^{-(y-1)^2/2\sigma^2} &= (1-2p)e^{-(y+1)^2/2\sigma^2} \\ y &= 0. \end{aligned}$$

   (c) When A is Rayleigh, the received signal constellation is the entire x-axis. In this case, the decision threshold will work out to be 0 as well.

6. (a) The BPSK case has been done in class in detail.

   (b) The LLR in this case will work out to

$$\log \frac{pe^{-(y+1)^2/2\sigma^2} + (1-p)e^{-(y-1)^2/2\sigma^2}}{pe^{-(y-1)^2/2\sigma^2} + (1-p)e^{-(y+1)^2/2\sigma^2}} = \log \frac{p + (1-p)e^{2y/\sigma^2}}{1 - p + pe^{2y/\sigma^2}}.$$

   (c) No solution provided yet.

7. Here is a simple method. Since $Y = AX + N$ and $X \in \{\pm 1\}$, $Y^2 = A^2 + 2AXN + N^2$. Taking expected values, $E[Y^2] = A^2 + \sigma^2$. Hence, a simple estimate for A is $\sqrt{E[Y^2] - \sigma^2}$ assuming A is positive. A MSE estimate can be found by using ideas from constrained complexity (1-tap) equalization.