Solutions to Problem Set 4

EE419: Digital Communication Systems

Check the solutions for possible bugs!

1. A basis for the signal set is $\{\phi(t) = \sqrt{1/T}, 0 \le t \le T\}$.
   (a) The received signal should be correlated with $\phi(t)$ and sampled at symbol rate for optimal processing.
   (b) The signal constellation is 1-dimensional with bit 0 at the origin and bit 1 at $A\sqrt{T}$. The average signal energy is therefore $A^2T/2$. Hence,
   $$\mathrm{SNR} = \frac{A^2T/2}{N_0/2} = \frac{A^2T}{N_0},$$
   assuming the 1-D noise variance is $\sigma^2 = N_0/2$. From the constellation, probability of symbol error is given by
   $$P_e = Q\left(\frac{A\sqrt{T}}{2\sigma}\right) = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right) = Q(\sqrt{\mathrm{SNR}/2}).$$
   (c) In this case (BPSK), SNR=$2A^2T/N_0$ and
   $$P_e = Q\left(\frac{2A\sqrt{T}}{2\sigma}\right) = Q(\sqrt{\mathrm{SNR}}).$$

2. (a) A basis for the signal set is $\{\phi_1(t) = \sqrt{2/T}\cos(2\pi t/T), \phi_2(t) = \sqrt{2/T}\cos(4\pi t/T)(0 \le t \le T)\}$. The received signal should be correlated with $\phi_1(t)$ and $\phi_2(t)$ and sampled at symbol rate for optimal processing.
   (b) The signal constellation is 2-D with bit 0 at $(\sqrt{T/2}, 0)$ and bit 1 at $(0, \sqrt{T/2})$. Hence,
   $$\mathrm{SNR} = \frac{T/2}{N_0} = \frac{T}{2N_0},$$
   assuming the 1-D noise variance is $\sigma^2 = N_0/2$. From the constellation, probability of symbol error is given by
   $$P_e = Q\left(\frac{\sqrt{T}}{2\sigma}\right) = Q(\sqrt{\mathrm{SNR}}).$$
   (c) In this case, the first basis signal can be taken to be $\phi_1(t) = \sqrt{2/T}\cos(\pi t/T), 0 \le t \le T$. The second basis element is to be found by Gram-Schmidt orthonormalization process as $\phi_2(t) = [s_2(t) - < s_2(t), \phi_1(t) > \phi_1(t)]_{\text{unit-norm}}$, where $s_2(t) = \cos(3\pi t/2T), 0 \le t \le T$. Since $< s_2(t), \phi_1(t) > = \dfrac{6\sqrt{2T}}{5\pi}$ (check this), we have
   $$\bar{\phi}_2(t) = s_2(t) - < s_2(t), \phi_1(t) > \phi_1(t) = \cos(3\pi t/2T) - \frac{12}{5\pi}\cos(\pi t/T).$$
   Since
   $$||\bar{\phi}_2(t)||^2 = ||s_2(t)||^2 - | < s_2(t), \phi_1(t) > |^2 = \frac{T}{2} - \frac{72T}{25\pi^2},$$

(check this) we have $\phi_2(t) = \bar{\phi}_2(t)/||\bar{\phi}_2(t)||$. The signal constellation is 2-D with bit 0 at $(\sqrt{T/2}, 0)$ and bit 1 at

$$(< s_2(t), \phi_1(t) >, ||\bar{\phi}_2(t)||) = \left(\frac{6\sqrt{2T}}{5\pi}, \sqrt{\frac{T}{2} - \frac{72T}{25\pi^2}} \right) \approx (0.54\sqrt{T}, 0.46\sqrt{T}).$$

The SNR is $\frac{T}{2N_0}$ and probability of error is given by

$$P_e = Q\left(\frac{\sqrt{(1 - \frac{12}{5\pi})T}}{2\sigma} \right) \approx Q(\sqrt{0.24\text{SNR}}).$$

3. There are four interior points each whose probability of correct decision is given by $P_1 = (1 - 2Q(d/2\sigma))^2$. There are four corner points whose probability of correct decision is given by $P_2(1 - Q(d/2\sigma))^2$. There are eight other points whose probability of correct decision is given by $P_3 = (1 - 2Q(d/2\sigma))(1 - Q(d/2\sigma))$. Hence, probability of symbol error is given by

$$\begin{aligned} P_e &= (1/4)(1 - P_1) + (1/4)(1 - P_2) + (1/2)(1 - P_3) \\ &= 3Q(d/2\sigma) - 2.25Q^2(d/2\sigma). \end{aligned}$$

   Generalization to 2^{2b}-QAM is easily done by counting the number of interior, corner and other points.

4. Refer standard DSP texts for a proof.

5. (a) Remember to square $H(f) = 1 - |f|/W, |f| < W$ and obtain the aliased spectrum. The plots are easy to do.

   (b) Is there any $1/T$ for which the folded spectrum satisfies the Nyquist criterion? The folded spectrum will not satisfy Nyquist exactly. Can you prove this?

   (c) The folded power spectral density is given by

$$S(e^{j\omega}) = \frac{1}{T} \sum_{m=-\infty}^{\infty} |H(f + \frac{m}{T})|^2$$

   with $\omega = 2\pi fT$. We evaluate $S(e^{j\omega})$ for $\omega \in [-\pi, \pi]$ or in $f \in [-1/2T, 1/2T]$. Finally, we find $\rho_h[k]$ as the inverse DTFT of $S(e^{j\omega})$.

   i. For $T = 1/2W$, $S(e^{j\omega}) = 2W(1 - |f|/W)^2$ for $|f| < W = 1/2T$ or

$$S(e^{j\omega}) = 2W(1 - |\omega|/\pi)^2, \quad |\omega| < \pi.$$

   Doing inverse DTFT, we get

$$\rho_h[k] = \frac{8W}{\pi k^2}.$$

   ii. For $T = 1/W$, $S(e^{j\omega}) = W\left((1 - |f|/W)^2 + (|f|/W)^2\right)$ for $|f| < W/2 = 1/2T$ or

$$S(e^{j\omega}) = W\left((1 - |\omega|/2\pi)^2 + (|\omega|/2\pi)^2\right), \quad |\omega| < \pi.$$

   Doing inverse DTFT, we get

$$\rho_h[k] = \frac{2W}{\pi k^2}.$$

iii. For $T = 2/W$, $S(e^{j\omega}) = W/2\left((1-|f|/W)^2 + (|f|/W)^2 + (1/2+|f|/W)^2\right)$ for $|f| < W/4 = 1/2T$ or

$$S(e^{j\omega}) = W/2\left((1-|\omega|/4\pi)^2 + (|\omega|/4\pi)^2 + (1/2+|\omega|/4\pi)^2\right), \quad |\omega| < \pi.$$

Doing inverse DTFT, we get

$$\rho_h[k] = \frac{W(1+0.5\cos(k\pi))}{4\pi k^2}.$$

6. The following two cases seem to work.
   (a) Transmit filter frequency response $\mathrm{rect}_{[-W/2,W/2]}(f)$ and symbol rate $W/2$.
   (b) Transmit filter frequency response $\mathrm{rect}_{[-W,W]}(f)$ and symbol rate W.

7. Transmit filter frequency response $\mathrm{rect}_{[-W/2,W/2]}(f)$ and symbol rate $W/2$. Surprisingly, transmit filter frequency response $\mathrm{rect}_{[-W,W]}(f)$ and symbol rate $W/2$ seems to work (check this).

8. (a) Since the phase response does not affect the folded spectrum, Nyquist criterion can be easily satisfied at the output of a matched filter for several symbol rates. However, the matched filter will have to implement the phase response $e^{-j\phi(f)}$.
   (b) If $\phi(f) = 2\pi f\tau$, the matched filter is simply a delay.
   (c) If $\phi(f)$ is non-linear, the implementation of the matched filter will have to meet the exact phase response requirement. Typically, one might do a Taylor series approximation in simple implementations.

9. Using $\cos\theta = (z+z^{-1})/2$ in $S_h(e^{j\theta})$, get $S_h(z)$ without any simplification. Use $u = (z+z^{-1})/2$ to get $S_h(u)$, which will be a quadratic in u. Solve to find two roots u_1 and u_2 for $S_h(u)$. Then use $u_i = (z+z^{-1})/2$ for $i = 1, 2$ to get the four roots of $S_h(z)$. Group the roots inside the unit circle and find $M(z)$.

10. Since there is no aliasing and the transmit filter response is flat, the folded spectrum is

$$S(e^{j2\pi fT}) = \sin^2(\frac{\pi f}{2W}), \ 0 \le |f| \le W.$$

Setting $\omega = 2\pi fT$ ans using $WT = 1/2$, we get

$$S(e^{j\omega}) = \sin^2(w/2) = 0.5(1-\cos(w)), \ |\omega| < \pi.$$

Converting to z-transform, we get

$$S(z) = 0.5(1-(z+z^{-1})/2) = 0.25(1-z^{-1})(1-z).$$

Hence, after a symbol-rate sampled WMF, the equivalent discrete-time channel is simply

$$M(z) = 1 - z^{-1}.$$