INFORMATION THEORY

PROBLEM SHEET - 7

① Given,

$$Y_i = X_i \oplus Z_i$$

where

$$Z_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{" } 1-p. \end{cases}$$

and Z_i are not independent.

Consider,

$$I(X_1, X_2 \dots X_n; Y_1, Y_2 \dots Y_n)$$

$$= H(X_1, X_2 \dots X_n) - H(X_1, X_2 \dots X_n / Y_1, Y_2 \dots Y_n)$$

$$= H(X_1, X_2 \dots X_n) - H(Z_1, Z_2 \dots Z_n / Y_1, Y_2 \dots Y_n)$$

$$\geq H(X_1, X_2 \dots X_n) - H(Z_1, Z_2 \dots Z_n)$$

$$\geq H(X_1, X_2 \dots X_n) - \sum_i H(Z_i)$$

$$= H(X_1, X_2 \dots X_n) - nH(p)$$

$$= n - nH(p)$$

If $X_1, X_2 \dots X_n$ are choosen i.i.d $\sim$ Bern$(1/2)$.

P.T.O.

The capacity of the channel with memory over 'n'-channel uses is

$$nC^{(n)} = \max_{P(x_1, x_2, \ldots x_n)} I(x_1, x_2 \ldots x_n ; y_1, y_2 \ldots y_n)$$

$$\geq I(x_1, x_2 \ldots x_n ; y_1, y_2 \ldots y_n)_{p(x_1, x_2 \ldots x_n) = \mathrm{Bern}(\frac{1}{2})}$$

$$\geq n(1 - H(P))$$

$$= nC.$$

Hence the channels with memory have higher capacity.

one can also observe the Intuitive explanation of the above result.

(2) $X = (x_1, x_2 \ldots x_n)$, $Y = (y_1, y_2 \ldots y_n)$.

Consider,

$$I(X ; Y) = H(Y) - H(Y/X)$$

$$= H(Y) - \sum_{i=1}^{n} H(y_i / y_1, y_2 \ldots y_{i-1}, X)$$

$$= H(Y) - \sum_{i=1}^{n} H(y_i / x_i)$$

Since by the definition of the channel, 'y_i' depends only on x_i and is conditionally independent of others.

$$I(X;Y) = H(Y) - \sum_{i=1}^{n} H(y_i/x_i)$$

$$\leq \sum_{i=1}^{n} H(y_i) - \sum_{i=1}^{n} H(y_i/x_i)$$

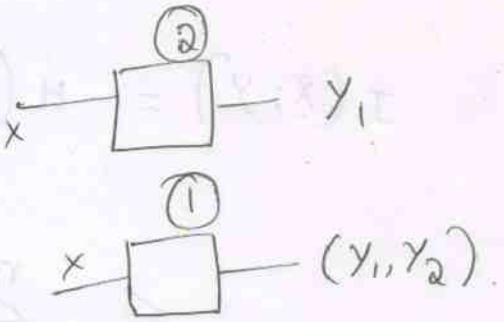
$$\leq \sum_{i=1}^{n} (1 - H(P_i))$$

with equality only when $x_1, x_2 \ldots x_n$ are choosen i.i.d.

Hence, $\max_{p(x)} I(X;Y) = \sum_{i=1}^{n} (1 - H(P_i))$.

(3) Addition of a row to the channel transition matrix is equivalent to adding a symol to the Input alphabet. Suppose there were (n) symbols and we make it '$n+1$' by adding a symbol. we can always choose not to transmit the new symbol. It means we can acheive the capacity C_n (with 'n' symbols only) always. Thus $C_{n+1} \geq C_n$

(4)



$$I(X; Y_1, Y_2)$$

$$= H(Y_1, Y_2) - H(Y_1, Y_2 / X)$$

$$= H(Y_1) + H(Y_2/Y_1) - H(Y_1/X) - H(Y_2/X)$$

[Indendentce of Y_1, Y_2]

$$= I(X; Y_1) + H(Y_2/Y_1) - H(Y_2/X) + H(Y_2) - H(Y_2)$$

$$= I(X; Y_1) + I(X; Y_2) - I(Y_1; Y_2)$$

$$= 2I(X; Y_1) - I(Y_1; Y_2)$$ [Y_1, Y_2 are Identically distributed].

(b) Since for any distribution, $I(Y_1; Y_2) \geq 0$, we can write,

$$I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1, Y_2) \leq 2I(X; Y_1).$$

Let C_1 be the capacity of $X \to (Y_1, Y_2)$ and $p^*(x)$ be the distributi[on] that acheives C_1. Then for $X \sim p^*(x)$,

$$C_1 = I_{p^*(x)}(X; Y_1, Y_2) \leq 2I_{p^*(x)}(X; Y_1) \leq 2C_2.$$

where C_2 is the capacity of the channel from X to Y_1.