EE6340: Information Theory
Problem Set 1

1. *Coin flips.* A fair coin is flipped until the first head occurs. Let X denote the number of flips required.

   (a) Find the entropy $H(X)$ in bits. The following expressions may be useful:

   $$\sum_{n=1}^{\infty} r^n = \frac{r}{(1-r)}, \qquad \sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}.$$

   (b) A random variable X is drawn according to this distribution. Find an "efficient" sequence of yes-no questions of the form, "Is X contained in the set S?" Compare $H(X)$ to the expected number of questions required to determine X.

2. *Entropy of functions of a random variable.* Let X be a discrete random variable. Show that the entropy of a function of X is less than or equal to the entropy of X by justifying the following steps:

   $$\begin{aligned} H(X, g(X)) &\stackrel{(a)}{=} H(X) + H(g(X)|X) \\ &\stackrel{(b)}{=} H(X); \end{aligned}$$

   $$\begin{aligned} H(X, g(X)) &\stackrel{(c)}{=} H(g(X)) + H(X|g(X)) \\ &\stackrel{(d)}{\geq} H(g(X). \end{aligned}$$

   Thus $H(g(X)) \leq H(X)$.

3. *Zero conditional entropy.* Show that if $H(Y|X) = 0$, then Y is a function of X, i.e., for all x with $p(x) > 0$, there is only one possible value of y with $p(x, y) > 0$

4. *World Series.* The World Series is a seven-game series that terminates as soon as either team wins four games. Let X be the random variable that represents the outcome of a World Series between teams A and B; possible values of X are AAAA, BABABAB, and BBBAAAA. Let Y be the number of games played, which ranges from 4 to 7. Assuming that A and B are equally matched and that the games are independent, calculate $H(X)$, $H(Y)$, $H(Y|X)$, and $H(X|Y)$.

5. *Example of joint entropy.* Let $p(x, y)$ be as shown in the table below.

   Find

   (a) $H(X), H(Y)$.

X \Y	0	1
0	1/3	1/3
1	0	1/3

(b) $H(X|Y), H(Y|X)$.

(c) $H(X,Y)$.

(d) $H(Y) - H(Y|X)$.

(e) $I(X;Y)$.

(f) Draw a Venn diagram for the quantities in (a) through (e).

6. *A measure of correlation.* Let X_1 and X_2 be identically distributed, but not necessarily independent. Let

$$\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)}.$$

(a) Show $\rho = \frac{I(X_1;X_2)}{H(X_1)}$.

(b) Show $0 \leq \rho \leq 1$.

(c) When is $\rho = 0$?

(d) When is $\rho = 1$?