EE6142 Quiz 1 solutions

① $T_\epsilon^{(n)}(X) = \{ x^n : |\pi(x/x^n) - p(x)| \le \epsilon\, p(x) \;\; \forall x \in \mathcal{X} \}$.

$(1-\epsilon)p(x) \le \pi(x/x^n) \le (1+\epsilon)p(x) \quad \forall x \in \mathcal{X}$.

(a) $0.9p(0) \le \pi(0/x^5) \le 1.1\, p(0)$ (or) $0.54 \le \pi(0/x^5) \le 0.66$ — ①

$0.9\, p(1) \le \pi(1/x^5) \le 1.1\, p(1)$ (or) $0.36 \le \pi(1/x^5) \le 0.44$ — ②

$$T_\epsilon^{(5)} = \left\{ x^5 : \begin{array}{l} 0.54 \le \pi(0/x^5) \le 0.66 \\ 0.36 \le \pi(1/x^5) \le 0.44 \end{array} \right\}$$

Sequences with 3 0's satisfy ①
Sequences with 2 1's satisfy ②

$T_\epsilon^{(5)} = \{$ all sequences with 3 0's and 2 1's $\}$ $\quad {}^5C_3 = 10$.

$$T_{0.1}^{(5)} = \{ 00011,\ 00110,\ 01100,\ 01010,\ 00101,\ 01001,\ 11000,\ 10100,\ 10010,\ 10001 \}$$

(b) $\epsilon = 0.6$

$$T_\epsilon^{(5)} = \left\{ x^5 : \begin{array}{l} 0.4p(0) \le \pi(0/x^5) \le 1.6p(0) \\ 0.4\, p(1) \le \pi(1/x^5) \le 1.6p(1) \end{array} \right\}$$

$$= \left\{ x^5 : \begin{array}{l} 0.24 \le \pi(0/x^5) \le 0.96 \\ 0.16 \le \pi(1/x^5) \le 0.64 \end{array} \right\}$$

First constraint is satisfied by sequences with 2, 3, or 4 0's.
Second constraint is satisfied by ——" —— 1, 2, or 3 1's.

$T_\epsilon^{(5)} = \{$ all sequences with 4 0's & one 1, 3 0's & 2 1's, 2 0's & 3 1's $\}$

$|T_\epsilon^{(5)}| = {}^5C_4 + {}^5C_3 + {}^5C_2 = \underline{25}$.

(c) $\epsilon = 0.4$

$$T_\epsilon^{(5)} = \left\{ x^5 : \begin{array}{l} 0.36 \le \pi(0/x^5) \le 0.84 \\ 0.24 \le \pi(1/x^5) \le 0.56 \end{array} \right\}$$

1st constraint is satisfied by sequences with 2, 3, 4 0's
2nd ——" —— with 2 1's. $\Rightarrow T_\epsilon^{(5)}$ <u>same as case (a)</u>

(2) Let $\mathcal{B}_n = \{(x^n, y^n, z^n) : (x^n, y^n) \in T_\epsilon^{(n)}(X,Y), (y^n, z^n) \in T_\epsilon^{(n)}(Y,Z)\}$

$|A_n| \leq |\mathcal{B}_n|$.

$$|\mathcal{B}_n| = \left| \bigcup_{y^n \in T_\epsilon^{(n)}(Y)} \{(x^n, y^n, z^n) : x^n \in T_\epsilon^{(n)}(X/y^n),\ z^n \in T_\epsilon^{(n)}(Z/y^n)\} \right|$$

$$\leq |T_\epsilon^{(n)}(Y)|\, |T_\epsilon^{(n)}(X/y^n)|\, |T_\epsilon^{(n)}(Z/y^n)|$$

$$\leq 2^{n(H(Y)+\delta(\epsilon))}\, 2^{n(H(X/Y)+\delta(\epsilon))}\, 2^{n(H(Z/Y)+\delta(\epsilon))}$$

$$= 2^{n(H(X,Y) + H(Z/Y) + 3\delta(\epsilon))}.$$

Similarly, let $\mathcal{C}_n = \{(x^n, y^n, z^n) : (y^n, z^n) \in T_\epsilon^{(n)}(Y,Z), (x^n, z^n) \in T_\epsilon^{(n)}(X,Z)\}$

$$|\mathcal{C}_n| \leq 2^{n(H(Y,Z) + H(X/Z) + 3\delta(\epsilon))}$$

$$|A_n|^2 \leq |\mathcal{B}_n||\mathcal{C}_n| = 2^{n(H(X,Y) + H(Y,Z) + \underbrace{H(Z/Y) + H(X/Z)}_{\leq H(X,Z)} + 6\delta(\epsilon))}$$

$$\leq 2^{n(H(X,Y) + H(Y,Z) + H(X,Z) + 6\delta(\epsilon))}$$

$$|A_n| \leq 2^{n(H(X,Y) + H(Y,Z) + H(X,Z) + 6\delta(\epsilon))/2}$$

(3)

$$R(D) = \min_{p(\hat{x}/x) : E[(X-\hat{X})^2] \leq D} I(X;\hat{X})$$

$$= \min_{p(\hat{x}/x) : E[(X-\hat{X})^2] \leq D} h(X) - h(X/\hat{X})$$

$$= h(X) - \max_{p(\hat{x}/x) : E[(X-\hat{X})^2] \leq D} h(X/\hat{X})$$

$$= h(X) - \max_{p(\hat{x}/x) : E[(X-\hat{X})^2] \leq D} h(X-\hat{X}/\hat{X})$$

$$\geq h(X) - \max_{p(\hat{x}/x) : E[(X-\hat{X})^2] \leq D} h(X-\hat{X})$$

$$\geq h(X) - \frac{1}{2}\log(2\pi e D).$$ ~~Equality iff $X - \hat{X} \sim WGN(D)$~~

(a) $I(X_1;Y|X_2=0) = I(X_1;0|X_2=0) = 0$.

$$\max_{p(x_1)} I(X_1;Y|X_2=0) = 0$$

$$I(X_1;Y|X_2=1) = I(X_1;X_1|X_2=1) = H(X_1)$$

$$\max_{p(x_1)} H(X_1) = 1.$$

(b)
$$\begin{aligned} I(X_1,X_2;Y) &= H(Y) - H(Y|X_1,X_2) \\ &= H(Y) \\ &\leq 1 \end{aligned}$$

Upper bound is achieved by $X_1 \sim \text{Bern}(\frac{1}{2})$, $X_2 \sim 1$ w.p. 1.

or

$X_2 \sim \text{Bern}(\frac{1}{2})$, $X_1 = 1$ w.p. 1

$$\max_{p(x_1)p(x_2)} I(X_1,X_2;Y) = 1.$$