Problem set 7

① $p_\theta(y) = \frac{\theta}{2} e^{-\theta|y|} \quad y \in \mathbb{R}.$

$$w(\theta) = \begin{cases} \frac{1}{\theta} & 1 \le \theta \le e \\ 0 & \text{else} \end{cases}$$

(a) $$w(\theta/y) = \frac{w(\theta)\, p_\theta(y)}{p(y)}$$

$$w(\theta) p_\theta(y) = \begin{cases} \frac{1}{2} e^{-\theta|y|} & 1 \le \theta \le e,\ y \in \mathbb{R}. \\ 0 & \text{else.} \end{cases}$$

$$p(y) = \int_1^e w(\theta) p_\theta(y)\, d\theta = \int_1^e \frac{1}{2} e^{-\theta|y|} d\theta = \left.\frac{e^{-\theta|y|}}{-2|y|}\right|_1^e = \frac{e^{-|y|} - e^{-e|y|}}{2|y|}$$

$$\hat{\theta}_{MAP}(y) = \arg\max_\theta\ w(\theta) p_\theta(y) \boxed{= 1.}$$

(b) $$\underbrace{\hat{\theta}_{MMSE}(y)}_{E[\Theta/Y=y]} = \int_1^e \theta\, w(\theta/y)\, d\theta = \frac{1}{p(y)} \int_1^e \frac{\theta}{2} e^{-\theta|y|} d\theta$$

$$= \frac{1}{p(y)} \int_1^e \frac{\theta}{2} \frac{d(e^{-\theta|y|})}{(-|y|)} = \frac{1}{p(y)} \left[\left.\frac{\theta e^{-\theta|y|}}{-2|y|}\right|_1^e - \int_1^e \frac{e^{-\theta|y|}}{(-|y|)} \frac{d\theta}{2} \right]$$

$$= \frac{1}{p(y)} \left[\frac{-e \cdot e^{-e|y|}}{2|y|} + \frac{e^{-|y|}}{2|y|} + \underbrace{\frac{1}{2|y|} \int_1^e e^{-\theta|y|} d\theta}_{\frac{p(y)}{|y|}} \right]$$

$$= \frac{1}{|y|} + \frac{e^{-|y|} - e \cdot e^{-e|y|}}{2|y|\, p(y)}.$$

$$= \boxed{\frac{1}{|y|} + \frac{e^{-|y|} - e \cdot e^{-e|y|}}{e^{-|y|} - e^{-e|y|}}}$$

(2) $Y = N + \theta S$

$P(S=1) = P(S=-1) = \frac{1}{2}$ $\quad$ N, θ, S independent.

$N \sim N(0,1)$

$$\Theta \sim w(\theta) = \begin{cases} k\, e^{\theta^2/2} & 0 \le \theta \le 1 \\ 0 & \text{else} \end{cases} \qquad k = \frac{1}{\int_0^1 e^{\theta^2/2} d\theta}$$

(a) $$w(\theta|y) = \frac{w(\theta) p_\theta(y)}{p(y)}$$

$$p_\theta(y) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{(y+\theta)^2}{2}}$$

$$= \frac{1}{2\sqrt{2\pi}} e^{\frac{-y^2-\theta^2+2y\theta}{2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{\frac{-y^2-\theta^2-2y\theta}{2}}$$

$$w(\theta) p_\theta(y) = \frac{k}{2\sqrt{2\pi}} e^{\frac{-y^2+2y\theta}{2}} + \frac{k}{2\sqrt{2\pi}} e^{\frac{-y^2-2y\theta}{2}} = \frac{k}{2\sqrt{2\pi}} e^{-y^2/2}\left[e^{y\theta} + e^{-y\theta}\right]$$

$$p(y) = \int_0^1 w(\theta) p_\theta(y)\, d\theta = \frac{k}{2\sqrt{2\pi}} e^{-\frac{y^2}{2}} \int_0^1 e^{y\theta} d\theta + \frac{k}{2\sqrt{2\pi}} e^{-y^2/2} \int_0^1 e^{-y\theta} d\theta.$$

$$= \frac{k}{2\sqrt{2\pi}} e^{-y^2/2} \left.\frac{e^{y\theta}}{y}\right|_0^1 + \frac{k}{2\sqrt{2\pi}} e^{-y^2/2} \left.\frac{e^{-y\theta}}{-y}\right|_0^1$$

$$= \frac{k}{2\sqrt{2\pi}} e^{-y^2/2}\left(\frac{e^y - 1}{y}\right) + \frac{k}{2\sqrt{2\pi}} e^{-y^2/2}\left(\frac{1-e^{-y}}{y}\right)$$

$$= \frac{k}{2\sqrt{2\pi}} e^{-y^2/2}\left[\frac{e^y - e^{-y}}{y}\right]$$

$$\hat{\theta}_{MMSE}(y) = E[\Theta | Y=y] = \int_0^1 \theta\, w(\theta|y)\, d\theta = \frac{\int_0^1 \theta\left(e^{y\theta} + e^{-y\theta}\right) d\theta}{\left(\frac{e^y - e^{-y}}{y}\right)}$$

$$\int_0^1 \theta e^{y\theta}\, d\theta = \int_0^1 \theta \, d\left(\frac{e^{y\theta}}{y}\right) = \frac{\theta e^{y\theta}}{y}\Big|_0^1 - \int_0^1 \frac{e^{y\theta}}{y}\, d\theta$$

$$= \frac{e^y}{y} - \frac{e^{y\theta}}{y^2}\Big|_0^1 = \frac{e^y}{y} - \frac{e^y - 1}{y^2}.$$

$$\int_0^1 \theta e^{-y\theta}\, d\theta = \int_0^1 \theta \,\frac{d(e^{-y\theta})}{-y} = \frac{\theta e^{-y\theta}}{-y}\Big|_0^1 + \int_0^1 \frac{e^{-y\theta}}{y}\, d\theta$$

$$= \frac{e^{-y}}{-y} + \frac{e^{-y\theta}}{-y^2}\Big|_0^1$$

$$= \frac{-e^{-y}}{y} + \frac{e^{-y}}{-y^2} \ldots = \frac{-e^{-y}}{y} + \frac{1 - e^{-y}}{y^2}.$$

$$\hat{\theta}_{MMSE}(y) = \frac{\frac{e^y - e^{-y}}{y} - \frac{e^y}{y^2} + \frac{1}{y^2} + \frac{1}{y^2} - \frac{e^{-y}}{y^2}}{\left(\frac{e^y - e^{-y}}{y}\right)}$$

$$= \frac{e^y - e^{-y} - \frac{e^y + e^{-y}}{y} + \frac{2}{y}}{e^y - e^{-y}}$$

$$= \boxed{1 - \frac{e^y + e^{-y}}{y(e^y - e^{-y})} + \frac{2}{y(e^y - e^{-y})}}$$

(b) $\hat{\theta}_{MAP}(y) = \arg\max_\theta \, w(\theta) p_\theta(y)$

$= \arg\max_\theta \left(e^{y\theta} + e^{-y\theta}\right)$

$\boxed{= \cancel{0}\, 1}$

at $\theta = 0$ $1 + 1 = 2.$

$y e^{y\theta} - y e^{-y\theta} \geq 0$ for $y > 0$.

$e^{y\theta} = e^{-y\theta}$

$\theta = 0.$

$y^2 e^{y\theta} + y^2 e^{-y\theta} > 0.$