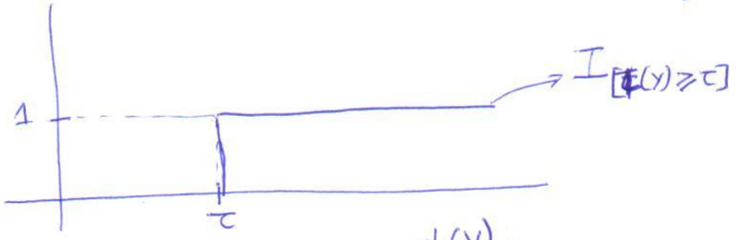
Solution to PS6.

(1)

$$P_e \le \pi_0 P_F + \pi_1 P_M$$

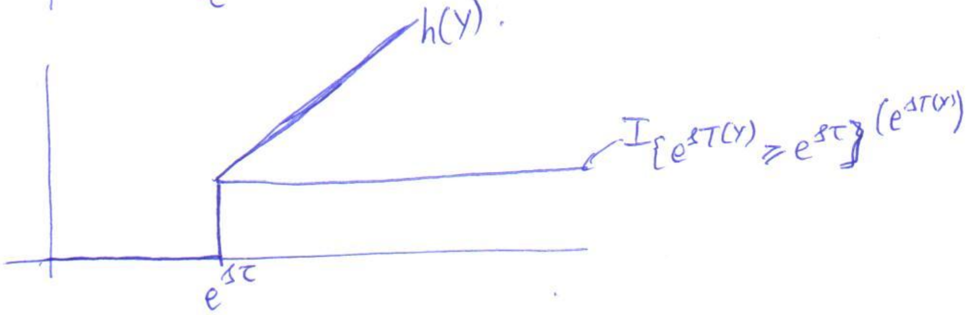
$$P_F \le P_0\left[T(Y) \ge \tau\right] = P_0\left[e^{sT(Y)} \ge e^{s\tau}\right] \quad \text{for } s \ge 0$$

$$= E_0\left[I_{[T(Y)\ge\tau]}(Y)\right]$$



$$\le E_0\left[h(Y)\right]$$

where $h(Y) = \begin{cases} \dfrac{e^{sT(Y)}}{e^{s\tau}} & \text{if } T(Y) \ge \tau \\ 0 & \text{else} \end{cases}$



$$P_F \le e^{-s\tau} \int_{\Gamma_1} e^{sT(Y)} p_0 \, d\mu = e^{-s\tau} \int_{\Gamma_1} L^s p_0 \, d\mu.$$

where $\Gamma_1 = \{T(Y) \ge \tau\}$.

$$P_M \le P_1\left[T(Y) \le \tau\right] = P_1\left[e^{tT(Y)} \le e^{t\tau}\right] \qquad t < 0$$

$$\le e^{-t\tau} \int_{\Gamma_0} L^t p_1 \, d\mu$$

$$= e^{-t\tau} \int_{\Gamma_0} L^{t+1} p_0 \, d\mu \qquad t < 0. \quad t+1 = s$$

$$= e^{-(s-1)\tau} \int_{\Gamma_0} L^s p_0 \, d\mu. \qquad s < 1$$

$$P_e \le \pi_0 e^{s\tau} \int_{\Gamma_1} L^s p_0 \, d\mu + \pi_1 e^{(1-s)\tau} \int_{\Gamma_0} L^s p_0 \, d\mu \qquad 0 < s < 1.$$

$$\int_{\Gamma_1} L^s p_0 \, d\mu = \int_{\Gamma} L^s p_0 \, d\mu - \int_{\Gamma_0} L^s p_0 \, d\mu$$

$$\Rightarrow P_e \le \pi_0 e^{-s\tau} \int_{\Gamma} L^s p_0 \, d\mu + (-\pi_0 e^{s\tau} + \pi_1 e^{(1-s)\tau}) \int_{\Gamma_0} L^s p_0 \, d\mu .$$

if $\pi_0 e^{-s\tau} \ge \pi_1 e^{(1-s)\tau}$ then

$$P_e \le \pi_0 e^{-s\tau} \int_{\Gamma} L^s p_0 \, d\mu = \pi_0 e^{-s\tau} e^{\mu_{T,0}(s)} . \quad (1)$$

if $\pi_1 e^{(1-s)\tau} > \pi_0 e^{s\tau}$

$$P_e \le \pi_1 e^{(1-s)\tau} \int_{\Gamma} L^s p_0 \, d\mu + (\pi_0 e^{s\tau} - \pi_1 e^{(1-s)\tau}) \int_{\Gamma_1} L^s p_0 \, d\mu$$

$$\le \pi_1 e^{(1-s)\tau} \int_{\Gamma} L^s p_0 \, d\mu = \pi_1 e^{(1-s)\tau} e^{\mu_{T,0}(s)} \quad (2)$$

(1) & (2)

$$\Rightarrow P_e \le \max\{\pi_0, \pi_1 e^{\tau}\} e^{(-s\tau + \mu_{T,0}(s))} \qquad 0 \le s \le 1$$

2) (Poor III F. 22)

$$H_0 : Y_k = N_k - S_k$$

$$H_1 : Y_k = N_k + S_k .$$

$N_1, N_2, \dots N_n$ are iid Laplacian. $s_1, s_2 \dots s_k$ are constants. s.t $s_k \geq \Delta > 0$

$$p_0(y) = \prod_{k=1}^{n} \frac{1}{2} e^{-|y_k + s_k|}$$

$$p_1(y) = \prod_{k=1}^{n} \frac{1}{2} e^{-|y_k - s_k|}$$

$$L(y) = \prod_{k=1}^{n} \left(\cancel{\tfrac{1}{2}} e^{-|y_k - s_k| + |y_k + s_k|} \right)$$

$$P_e \leq \max \{ \pi_0 , \pi_1 e^{\tau} \} \exp \{ \mu_{T,0}(s) - s\tau \} \qquad 0 \leq s \leq 1$$

$$\mu_{T,0}(s) = \log \left(\int_{\Gamma} L^s p_0 \, d\mu \right)$$

$$\int_{\Gamma} L^s p_0(\underline{y}) \, d\underline{y} = \prod_{k=1}^{n} \left(\int_{-\infty}^{\infty} \frac{1}{2} e^{-|y_k - s_k| s + |y_k + s_k| s} \times e^{-|y_k + s_k|} dy_k \right)$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} e^{-(s|y_k - s_k| + (1-s)|y_k + s_k|)} dy_k$$

$$= \frac{1}{2} \left[\int_{-\infty}^{-s_k} e^{-(s(s_k - y_k) + (1-s)(-y_k - s_k))} dy_k + \int_{-s_k}^{s_k} e^{-(s(s_k - y_k) + (1-s)(y_k + s_k))} dy_k \right.$$

$$\left. + \int_{s_k}^{\infty} e^{-(s(y_k - s_k) + (1-s)(y_k + s_k))} dy_k \right]$$

$$= \frac{1}{2} \left[\int_{-\infty}^{-s_k} e^{-(ss_k - \cancel{sy_k} - y_k - s_k + \cancel{sy_k} + ss_k)} dy_k + \int_{-s_k}^{s_k} e^{-(ss_k - sy_k + y_k + s_k - \cancel{ss_k} - sy_k)} dy_k \right.$$

$$\left. + \int_{s_k}^{\infty} e^{-(sy_k - ss_k + y_k + s_k - sy_k - ss_k)} dy_k \right]$$

$$= \frac{1}{2}\left[e^{\delta_k - 2\delta\delta_k} \times e^{y_k}\Big|_{-\infty}^{-\delta_k} + e^{-\delta_k} \frac{e^{-(1-2\delta)y_k}}{-(1-2\delta)}\Big|_{-\delta_k}^{\delta_k} + e^{-\delta_k + 2\delta\delta_k} \frac{e^{-y_k}}{-1}\Big|_{\delta_k}^{\infty} \right]$$

$$= \frac{1}{2}\left[e^{-2\delta\delta_k} + e^{-\delta_k} \frac{\left(e^{+\delta_k} e^{-2\delta\delta_k} - e^{-\delta_k} e^{2\delta\delta_k}\right)}{1-2\delta} + e^{-2\delta_k(1-\delta)} \right]$$

$$= \frac{1}{2}\left[e^{-2\delta\delta_k} + \frac{1}{(1-2\delta)}\left(e^{-2\delta\delta_k} - e^{-2(1-\delta)\delta_k} \right) + e^{-2(1-\delta)\delta_k} \right]$$

~~Let $\delta = \frac{1}{4}$~~

$$= \frac{1}{2}\left[\cancel{e^{-\frac{\delta_k}{2}}} + \frac{1}{2}\left(\cancel{e^{-\frac{\delta_k}{2}}} - e^{-\frac{3}{2}\delta_k} \right) + \cancel{e^{-\frac{3}{2}\delta_k}} \right]$$

$$= \cancel{\frac{3}{4} e^{-\frac{\delta_k}{2}}} + \cancel{\frac{1}{4} e^{-\frac{3}{2}\delta_k}}$$

$$= \frac{1}{2}\left[\frac{e^{-2\delta\delta_k}(2-2\delta) - e^{-2(1-\delta)\delta_k}(1-1+2\delta)}{1-2\delta} \right]$$

$$= \frac{1}{2}\left[\frac{\cancel{2(1-\delta}\ 2(1-\delta)\, e^{-2\delta\delta_k} - 2\delta\, e^{-2(1-\delta)\delta_k}}{1-2\delta} \right]$$

for $\delta = \frac{1}{4}$ we have.

$$= \frac{3}{2} e^{-\frac{\delta_k}{2}} - \frac{1}{2} \cancel{e^{-\delta_k}}\, e^{-\frac{3}{2}\delta_k} \qquad > 0 \quad \text{for } \delta_k > 0$$

$$\cancel{\leq \frac{3}{2} e^{-\frac{\delta_k}{2}} + \frac{1}{2} e^{-\frac{3}{2}\delta_k}}$$

~~Taking log we get~~

$$= \frac{3}{2} e^{-\frac{s_k}{2}} - \frac{1}{2} e^{-\frac{3}{2} s_k}. \quad -(1)$$

Diff w.r.t s_k

we get

$$\frac{3}{2} \times \left(\frac{-1}{2}\right) e^{-\frac{s_k}{2}} + \frac{1}{2} \times \frac{3}{2} e^{-\frac{3}{2} s_k}$$

$$= \frac{3}{2} \left\{ e^{-\frac{3}{2} s_k} - e^{-\frac{s_k}{2}} \right\} < 0$$

$\Rightarrow$ (1) is a decreasing function of s_k.

at $s_k = 0$

$$(1) = \frac{3}{2} - \frac{1}{2} = 1$$

$\Rightarrow$ (1) $\leq 1 \quad \forall s_k > 0$

Now

$$\int_\Gamma L^s p_0(y)\, dy = \prod_{k=1}^{n} \left(\frac{3}{2} e^{-\frac{s_k}{2}} - \frac{1}{2} e^{-\frac{3}{2} s_k}\right) \Bigg|_{s = \frac{1}{4}}$$

$$\leq \prod_{k=1}^{n} \left(\frac{3}{2} e^{-\frac{\Delta}{2}} - \frac{1}{2} e^{-\frac{3\Delta}{2}}\right)$$

$\because$ (1) is a decreasing fn. and $s_k > \Delta \quad \forall k$.

$$= \prod_{k=1}^{n} \alpha$$

$$= \alpha^n \quad \text{where } \alpha < 1$$

Now

$$P_e \leq \text{Max}\{\Pi_0, \Pi_1 e^{\tau}\} e^{-\frac{\tau}{4}} \alpha^n$$

$$\lim_{n \to \infty} P_e \leq 0 \qquad \because \alpha < 1$$

4.). iid Bernoulli observations $Y_1, Y_2, \ldots$

$H_0: \quad P(Y_k=1) = 1 - P(Y_k=0) = 1/3.$

$H_1: \quad P(Y_k=1) = 1 - P(Y_k=0) = 2/3.$

a). $p^* = \max(P_F, P_M) = 0.01$

$\alpha = P_F$

$\gamma = P_M$

$\Rightarrow \max(\alpha, \gamma) = 0.01$

From Wald's approximation,

$A \cong \dfrac{\gamma}{1-\alpha}.$

$B \cong \dfrac{1-\gamma}{\alpha}.$

$\Rightarrow A \leq \dfrac{0.01}{0.99} \Rightarrow A \leq 1/99.$

$B \geq \dfrac{0.99}{0.01} \Rightarrow B \geq 99.$

If we choose $A \leq 1/99$ and $B \geq 99$, we can always have $p^* = 0.01$.

$$E(N|H_0) = \frac{1}{\mu_0}\left[(1-\alpha)\log\left(\frac{\gamma}{1-\alpha}\right) + \alpha\log\left(\frac{1-\gamma}{\alpha}\right)\right]$$

$$E(N|H_1) = \frac{1}{\mu_1}\left[\gamma\log\left(\frac{\gamma}{1-\alpha}\right) + (1-\gamma)\log\frac{1-\gamma}{\alpha}\right].$$

$$\mu_j = E\left\{ \log\left(\frac{P_1(Y_1)}{P_0(Y_1)}\right) \Big| H_j \right\}.$$

$$\mu_0 = \log\left(\frac{2/3}{1/3}\right) \times 1/3 + \log\left(\frac{1/3}{2/3}\right) \times \frac{2}{3}.$$

$$= \frac{1}{3}\log 2 - \frac{2}{3}\log 2.$$

$$= -\frac{1}{3}\log 2.$$

$$\mu_1 = \log\left(\frac{2/3}{1/3}\right) \times \frac{2}{3} + \log\left(\frac{1/3}{2/3}\right) \times \frac{1}{3}.$$

$$= \frac{2}{3}\log 2 - \frac{1}{3}\log 2$$

$$= \frac{1}{3}\log 2.$$

Choose $A = 1/99$, $B = 99$. $\Rightarrow \alpha = \gamma = 0.01$.

Then $E\{N|H_0\} = \dfrac{-3}{\log 2}\left[0.99 \log \dfrac{0.01}{0.99} + 0.01 \log \dfrac{0.99}{0.11}\right]$

$$= \frac{-3}{\log 2.}\left[0.99 \log \frac{1}{99} + 0.01 \log 99\right]$$

$$= \frac{-3}{\log 2}\left[-0.99 \log 99 + 0.01 \log 99\right]$$

$$= \frac{3 \times 0.98}{\log 2} \log 99 = \frac{2.94 \log 99}{\log 2}.$$

$$E\{N|H_1\} = \frac{3}{\log 2}\left[0.01 \log \frac{0.01}{0.99} + 0.99 \log \frac{0.99}{0.01}\right].$$

$$\Rightarrow E\{N|H_1\} = \frac{3}{\log 2}\cdot\left[-0.01 \log 99 + 0.99 \log 99\right]$$

$$E\{N|H_1\} = \frac{2.94 \log 99}{\log 2}.$$

The calculate values of $E\{N|H_0\}$ & $E\{N|H_1\}$ will vary depending on the exact α, γ we choose and in effect the A & B chosen.