Problem Set - 5

1) $\cancel{Y_k} = S_k + \theta N_1$

$Y_k = N_k + \theta S_k \;;\; k = 1, 2, \dots n. \quad N \sim \mathcal{N}(0, I)$

$S_1, S_2, \dots S_n$ iid $\sim U\{+1, -1\}$.

a) $H_0 : \theta = 0$ versus. $H_1 : \theta = A$.

when $\theta = 0$

$y_k = N_k$,

$$\Rightarrow p_0(\underline{y}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}}$$

when $\theta = A$

$y_k = N_k + A$ w.p $\frac{1}{2}$

$N_k - A$ w.p $\frac{1}{2}$.

$$\Rightarrow p_1(y) = \prod_{i=1}^{n} \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i - A)^2}{2}} + \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i + A)^2}{2}}$$

$$L(y) = \frac{p_1(y)}{p_0(y)}$$

$$= \prod_{i=1}^{n} \left(\frac{\cancel{\frac{1}{\sqrt{2\pi}}} \frac{1}{2} e^{-\frac{(y_i - A)^2}{2}} + \cancel{\frac{1}{\sqrt{2\pi}}} \frac{1}{2} e^{-\frac{(y_i + A)^2}{2}}}{\cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{y_i^2}{2}}} \right)$$

$$= \prod_{i=1}^{n} e^{-\frac{A^2}{2}} \left(\frac{1}{2} e^{+Ay_i} + \frac{1}{2} e^{-Ay_i} \right)$$

$$= \prod_{i=1}^{n} e^{-\frac{A^2}{2}} \cosh(Ay_i)$$

b) $n = 1 \;;\; P_{FA} = \alpha$.

$$\delta = \begin{cases} 1 & \text{if } L(y) \geq \tau \\ 0 & \text{if } L(y) < \tau \end{cases}$$

$$L(y) = e^{-\frac{A^2}{2}} \cosh(Ay)$$

$L(y) > \tau$

$\Rightarrow \quad \cosh(Ay) > e^{+\frac{A^2}{2}}\tau$

$Ay > \cosh^{-1}\left(e^{+\frac{A^2}{2}}\tau\right)$ when $Ay > 0$.

$Ay < -\cosh^{-1}\left(e^{+\frac{A^2}{2}}\tau\right)$

Here $\quad y > \frac{1}{A}\cosh^{-1}\left(e^{\frac{A^2}{2}}\tau\right)$

$y < \frac{-1}{A}\cosh^{-1}\left(e^{+\frac{A^2}{2}}\tau\right)$

The decision rule is the same for both +ve and -ve values of A.

Let $\frac{1}{A}\cosh^{-1}\left(e^{\frac{A^2}{2}}\tau\right) = \tau'$. And τ' can be written solely in terms of α.

$P_{FA} =$

$\Rightarrow$ UMP exists.

$$P_{FA} = \int_{\tau'}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy + \int_{-\infty}^{-\tau'} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

$$= 2Q(\tau')$$

$$\tau' = Q^{-1}\left(\frac{\alpha}{2}\right).$$

Now $\quad P_D = \left\{\frac{1}{2}\int_{\tau'}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}} dy + \frac{1}{2}\int_{\tau'}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}}\right\} \times 2.$

$$= Q(\tau' - A) + Q(\tau' + A)$$

$$P_D = Q\left(Q^{-1}\left(\frac{\alpha}{2}\right) - A\right) + Q\left(Q^{-1}\left(\frac{\alpha}{2}\right) + A\right)$$

c) From part (b) UMP exists for $n=1$

For $n > 1$

$L(y) > \tau \quad \Rightarrow \quad \prod_{i=1}^{n} e^{-\frac{A^2}{2}} \cosh(Ay_i) > \tau$

$$\prod_{i=1}^{n} \cosh(A y_i) > e^{\frac{A^2}{2}} \hat{c}$$

Let us find the region for a fixed y_j

we have.

$$\cosh(A y_j) > \frac{e^{-\frac{A^2}{2}} \hat{c}}{\prod_{i \neq j} \cosh(A y_i)}$$

By the same argument made for $n=1$ we can conclude that UMP exists.

2) $$Y_k = \theta^{1/2} s_k R_k + N_k \quad ; \quad N_i, R_i \text{ iid} \sim N(0,1)$$

a) $H_0 : \theta = 0$

vs.

$H_1 : \theta = A$

When $\theta = 0$

$Y_k = N_k$.

$$\Rightarrow p_0(y) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{- y_i^2}$$

When $\theta = A$

$y_k = \sqrt{A}\, s_k R_k + N_k$.

$E[y_k] = 0$

$\mathrm{Var}(y_k) = A s_k^2 + 1$

$$\Rightarrow C = E[y_k y_k^*]$$

$$= \begin{bmatrix} A s_1^2+1 & & & 0 \\ & A s_2^2+1 & & \\ & & \ddots & \\ 0 & & & A s_n^2+1 \end{bmatrix}$$

$$p_1(y) = \left(\frac{1}{2\pi}\right)^{\frac{N}{2}} \frac{1}{(\det(C))^{1/2}} e^{-\frac{1}{2} y^T C^{-1} y}$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{N}{2}} \frac{1}{(\det(C))^{1/2}} e^{-\frac{1}{2} \sum_{k=1}^{n} (A s_k^2 + 1)\, y_k^2}$$

$$L(y) = \frac{p_1(y)}{p_0(y)} = \frac{e^{-\frac{1}{2} \sum_{k=1}^{n} (A s_k^2 y_k^2 + y_k^2)}}{(\det(C))^{1/2}\, e^{-\sum_{k=1}^{N} y_k^2}}$$

$L(y) > \hat{c}$

$\Rightarrow \quad -\frac{1}{2}$

$$p_1(y) = \left(\frac{1}{2\pi}\right)^{\frac{N}{2}} \frac{1}{(\det(C))^{1/2}} e^{-\frac{1}{2}\sum_{k=1}^{n} \frac{y_k^2}{As_k^2+1}}$$

$$L(y) = \frac{p_1(y)}{p_0(y)} = \frac{1}{(\det(C))^{1/2}} \frac{e^{-\frac{1}{2}\sum_{k=1}^{n}\frac{y_k^2}{As_k^2+1}}}{e^{-\frac{1}{2}\sum_{k=1}^{n} y_k^2}}$$

$$= \frac{1}{(\det(C))^{1/2}} e^{-\frac{1}{2}\sum_{k=1}^{n}\left(\frac{y_k^2 - (As_k^2+1)y_k^2}{As_k^2+1}\right)}$$

$$= \frac{1}{\sqrt{\det(C)}} e^{+\frac{1}{2}\sum_{k=1}^{n}\left(\frac{Ay_k^2 s_k^2}{As_k^2+1}\right)}$$

$$L(y) > \tau$$

$$\Rightarrow \sum_{k=1}^{n}\left(\frac{Ay_k^2 s_k^2}{1+As_k^2}\right) > \tau' \quad (1) \quad \text{where } \tau' = 2\log\left(\sqrt{\det(C)}\,\tau\right)$$

b) $H_0: \theta = 0$

$H_1: \theta > 0$

if $s_k = s \ \forall k$.

then, we have,

$$\frac{As^2}{1+As^2} \sum_{k=1}^{n} y_k^2 > \tau'$$

(or) $$\sum_{k=1}^{n} y_k^2 > \tau'' \quad \text{where } \tau'' = \frac{\tau'(1+As^2)}{As^2}$$

~~Here~~ ~~τ'' cor~~ Here the decision rule is clearly the same for $A > 0$.

(ie) UMP exists when $s_1 = s_2 = \cdots = s_n = s$.

Locally optimum detector.

From ① we have.

$\delta = 1$ when $\displaystyle\sum_{k=1}^{n} \frac{\partial}{\partial A}\left(\frac{A y_k^2 s_k^2}{1+A s_k^2}\right)\Bigg|_{A=0} \geq \tau'$

$$\Rightarrow \sum_{k=1}^{n}\left(\frac{y_k^2 s_k^2 (1+A s_k^2) - A y_k^2 s_k^2 s_k^2}{(1+A s_k^2)^2}\right)\Bigg|_{A=0} \geq \tau'$$

$$\Rightarrow \sum_{k=1}^{n} y_k^2 s_k^2 \geq \tau'$$

LOD :

$$\delta(y) = \begin{cases} 1 & \text{if } \sum_{k=1}^{n} y_k^2 s_k^2 \geq \tau' \\ 0 & \text{if o.w.} \end{cases}$$

3)

$H_0 : Y_k = N_k.$ $\quad N_1, N_2 \ldots N_n$ are iid $\sim N(0, \sigma^2)$

$H_1 : Y_k = N_k + \Theta S_k.$ $\quad \Theta \sim N(\mu, \nu^2)$

$s^T s = 1$

$$p_0(\underline{y}) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\sum_{k=1}^{n} \frac{y_k^2}{2}}$$

$H_1 : Y_k = N_k + \Theta S_k.$

$E[Y_k] = \mu s_k$

$$E[Y_k Y_l] = (N_k + \Theta S_k)(N_l + \Theta S_l)$$

$$= \begin{cases} \nu^2 s_k s_l & l \neq k. \\ \sigma^2 + \nu^2 s_k^2 & l = k. \end{cases}$$

$$\Rightarrow E[Y Y^T] = \sigma^2 I + \nu^2 \underline{s}\,\underline{s}^T$$

$\Rightarrow$ under H_1, $\underline{Y}$ is gaussian with mean $\mu \underline{s}$ and $C = \sigma^2 I + \vartheta^2 s s^T$

$$C = \sigma^2 I + \vartheta^2 s s^T$$

$$C^{-1} = (\sigma^2 I + \vartheta^2 s s^T)^{-1}$$

By applying Matrix inversion lemma.

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1} D A^{-1}$$

we have

$$C^{-1} = \frac{I}{\sigma^2} - \frac{s s^T}{\sigma^4 \left(\frac{1}{\vartheta^2} + \frac{s^T s}{\sigma^2} \right)}$$

$$= \frac{1}{\sigma^2} \left(I - \frac{s s^T}{\frac{\sigma^2}{\vartheta^2} + s^T s} \right)$$

$$LLR = \frac{p_1(y)}{p_0(y)}$$

$$= \frac{\left(\frac{1}{2\pi}\right)^{\frac{N}{2}} \frac{1}{\sqrt{\det(C)}} e^{-\frac{1}{2}(\underline{y} - \mu \underline{s})^T C^{-1} (\underline{y} - \mu \underline{s})}}{\left(\frac{1}{2\pi}\right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2} \underline{y}^T \underline{y}}}$$

$LLR > \tau$ $\qquad\qquad$ $\tau' = \sqrt{\det C}\, \tau$

$\Rightarrow$ $\log LLR > \log \tau$

$$\Rightarrow -\frac{1}{2\sigma^2}(y - \mu s)^T \left(I - \frac{s s^T}{\frac{\sigma^2}{\vartheta^2} + s^T s} \right) (y - \mu s) + \frac{1}{2\sigma^2} y^T y > \tau'$$

$$\Rightarrow \frac{-1}{2\sigma^2}\left(\cancel{y^T y} + \mu^2 s^T s - 2\mu \underline{s}^T \underline{y} - \frac{(y^T s - \mu s^T s)(s^T y - \mu s^T s)}{\frac{\sigma^2}{\vartheta^2} + s^T s} - \cancel{y^T y} \right) > \tau'$$

$\because s^T s = 1$

we have.

$$\Rightarrow 2\mu s^T y + \frac{|s^T y - \mu|^2}{\frac{\sigma^2}{\vartheta^2} + 1} > \tau''$$

$$\tau'' = 2\sigma^2 \sqrt{\det C}\, \tau + \mu^2.$$

$$2\mu s^T y\left(\frac{\sigma^2}{\vartheta^2}+1\right) + |s^T y|^2 + \mu^2 - 2\mu s^T y > \varepsilon''$$

$$\Rightarrow \quad 2\mu s^T y \frac{\sigma^2}{\vartheta^2} + |s^T y|^2 > \varepsilon''$$

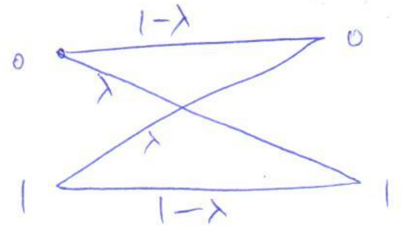
ε'_x

$$\Rightarrow \quad \mu s^T y + \frac{\vartheta^2}{\sigma^2 2} |s^T y|^2 > \varepsilon'$$

Here $\varepsilon' = \frac{(2\sigma^2\sqrt{\det C} - 2\mu^2)\,\vartheta^2}{2\sigma^2}$.

$$\Rightarrow \quad \Gamma_1 = \left\{ \mu s^T y + \frac{\vartheta^2}{2\sigma^2} |s^T y|^2 > \varepsilon' \right\}$$

4)



Equal priors $\Rightarrow$ $\tau = 1$.

$$L(y) = \begin{cases} \dfrac{\lambda}{1-\lambda} & y=0 \\[2ex] \dfrac{1-\lambda}{\lambda} & y=1. \end{cases}$$

Let $\lambda < \frac{1}{2}$.

$y=0$, $\dfrac{\lambda}{1-\lambda} < 1 \Rightarrow$ decision is H_0

$y=1$, $\dfrac{1-\lambda}{\lambda} > 1 \Rightarrow$ decision is H_1.

optimal decision: "accept y".

$\Rightarrow$ $\boxed{\text{Prob. of error} = \lambda.}$

Under H_0, $$L(Y) = \begin{cases} \dfrac{\lambda}{1-\lambda} & \text{with prob } (1-\lambda) \\[2ex] \dfrac{1-\lambda}{\lambda} & \text{with prob } \lambda. \end{cases}$$

$$P_e \le \max\{\pi_0, \pi_1 e^{\tau}\} \exp\{\mu_{T,0}(s) - s\tau\}$$

$$\pi_0 = \pi_1 = \tfrac{1}{2}, \quad \tau = \log 1 = 0$$

$$P_e \le \tfrac{1}{2} \exp\{\mu_{T,0}(s) - s\}.$$

$$\mu_{T,0}(s) = \log E\left[e^{s \log L(Y)} / H_0\right] = \log E\left[(L(Y))^s / H_0\right]$$
$$= \log\left[(1-\lambda)\cdot\left(\frac{\lambda}{1-\lambda}\right)^s + \lambda\cdot\left(\frac{1-\lambda}{\lambda}\right)^s\right]$$

$\Rightarrow P_e \leq \text{Max}\{\pi_0, \pi_1\} \left[(1-\lambda)\left(\frac{\lambda}{1-\lambda}\right)^s + \lambda\left(\frac{1-\lambda}{\lambda}\right)^s\right]$

$$= \frac{1}{2}\left[(1-\lambda)\left(\frac{\lambda}{1-\lambda}\right)^s + \lambda\left(\frac{1-\lambda}{\lambda}\right)^s\right] \quad (1)$$

Diff (1) w.r.t s.

we get,

$$\frac{1}{2}\left\{(1-\lambda)\left(\frac{\lambda}{1-\lambda}\right)^s \ln\left(\frac{\lambda}{1-\lambda}\right) + \lambda\left(\frac{1-\lambda}{\lambda}\right)^s \ln\left(\frac{1-\lambda}{\lambda}\right)\right\} = 0$$

$$\Rightarrow (1-\lambda)\left(\frac{\lambda}{1-\lambda}\right)^s = \left(\frac{1-\lambda}{\lambda}\right)^s \lambda$$

$$\Rightarrow (\lambda)^{2s-1} = (1-\lambda)^{2s-1}$$

$$\Rightarrow 2s-1 = 0 \qquad s = \frac{1}{2}$$

$\therefore$ Sub $s = \frac{1}{2}$ in (1) we get

$$P_e \leq \frac{1}{2}\left[(1-\lambda)\sqrt{\frac{\lambda}{1-\lambda}} + \lambda\sqrt{\frac{1-\lambda}{\lambda}}\right]$$

$$= \sqrt{\lambda(1-\lambda)}$$