Solutions to PS 4

1.) $P_{\theta}(y) = \begin{cases} \theta e^{-\theta y} & , y \geq 0 \\ 0 & , y < 0 \end{cases}$

$$w(\theta) = \begin{cases} \alpha e^{-\alpha \theta} & , \theta \geq 0 \\ 0 & , \theta < 0 \end{cases}$$

$$H_0 : \theta \in (0, \beta) \triangleq \Lambda_0$$

$$H_1 : \theta \in (\beta, \infty) \triangleq \Lambda_1$$

$$\frac{P_1(y)}{P_0(y)} > \frac{\pi_0}{\pi_1}$$

$$P_1(y) = \int_{\theta=\beta}^{\infty} \theta e^{-\theta y} \cdot \alpha e^{-\alpha\theta} \, d\theta$$

$$= \alpha \int_{\theta=\beta}^{\infty} \theta e^{-\theta(y+\alpha)} \, d\theta$$

$$= \alpha \left[\theta \frac{e^{-\theta(y+\alpha)}}{-(y+\alpha)} - \int \frac{e^{-\theta(y+\alpha)}}{-(y+\alpha)} \right]_{\theta=\beta}^{\infty}$$

$$= \alpha \left[\frac{-\theta e^{-\theta(y+\alpha)}}{(y+\alpha)} - \frac{e^{-\theta(y+\alpha)}}{(y+\alpha)^2} \right]_{\theta=\beta}^{\infty} = \alpha \left[\frac{+\beta e^{-\beta(y+\alpha)}}{(y+\alpha)} + \frac{e^{-\beta(y+\alpha)}}{(y+\alpha)^2} \right]$$

$$P_1(y) = \alpha e^{-\beta(y+\alpha)} \left[\frac{+\beta}{(y+\alpha)} + \frac{1}{(y+\alpha)^2} \right].$$

$$P_0(y) = \int_{\theta=0}^{\beta} \theta e^{-\theta y} \alpha e^{-\alpha\theta} \, d\theta.$$

$$= \alpha \left[\frac{-\beta e^{-\beta(y+\alpha)}}{(y+\alpha)} - \frac{e^{-\beta(y+\alpha)}}{(y+\alpha)^2} + \frac{1}{(y+\alpha)^2} \right].$$

$$\pi_1 = \int_{\theta=\beta}^{\alpha} \alpha e^{-\alpha\theta} \, d\theta = \alpha \left[\frac{e^{-\alpha\theta}}{-\alpha} \right]_{\beta}^{\infty}$$

$$= \cancel{\alpha e^{-\alpha\beta}} \; \alpha \left[0 - \frac{e^{-\alpha\beta}}{-\alpha} \right].$$

$$= \alpha \frac{e^{-\alpha\beta}}{\alpha} = e^{-\alpha\beta}$$

$$\pi_0 = \int_{\theta=0}^{\beta} \alpha e^{-\alpha\theta} \, d\theta = \alpha \left[\frac{e^{-\alpha\theta}}{-\alpha} \right]_{\theta=0}^{\beta}$$

$$= \alpha \left[\frac{e^{-\alpha\beta}}{-\alpha} - \frac{1}{-\alpha} \right].$$

$$= \left[1 - e^{-\alpha\beta} \right].$$

yes rule:

$$\Gamma_1 \text{ if } \frac{\alpha e^{-\beta(y+\alpha)}\left[\frac{\beta}{(y+\alpha)} + \frac{1}{(y+\alpha)^2}\right]}{\alpha\left[\frac{1}{(y+\alpha)^2} - \frac{e^{-\beta(y+\alpha)}}{(y+\alpha)^2} - \frac{\beta e^{-\beta(y+\alpha)}}{(y+\alpha)}\right]} > \frac{e^{-\alpha\beta}}{1-e^{-\alpha\beta}}.$$

$$\Rightarrow \Gamma_1 \text{ if. } \alpha e^{-\beta(y+\alpha)}\left[\frac{\beta}{(y+\alpha)} + \frac{1}{(y+\alpha)^2}\right] > \frac{\alpha e^{-\alpha\beta}}{(y+\alpha)^2}.$$

Γ_0 if otherwise.

Minimum Bayes risk. $\int_{\Gamma_1} \pi_0 P_0(y)dy + \int_{\Gamma_0} \pi_1 P_1(y)dy$.

2.). a.). The LMP test is

$$\tilde{\delta}_{lo}(y) = \begin{cases} 1 & \text{if } \left.\frac{\partial P_\theta(y)}{\partial\theta}\right|_{\theta=0} > \eta P_0(y). \\ \gamma & \text{if } \left.\frac{\partial P_\theta(y)}{\partial\theta}\right|_{\theta=0} = \eta P_0(y). \\ 0 & \text{if } \left.\frac{\partial P_\theta(y)}{\partial\theta}\right|_{\theta=0} < \eta P_0(y) \end{cases}$$

we have
$$\frac{\left.\frac{\partial P_\theta(y)}{\partial \theta}\right|_{\theta=0}}{P_0(y)} = \operatorname{sgn}(y).$$

thus.
$$\tilde{\delta}_{l_0}(y) = \begin{cases} 1 & \text{if } \operatorname{sgn}(y) > n \\ \gamma & \text{if } \operatorname{sgn}(y) = n. \\ 0 & \text{if } \operatorname{sgn}(y) < n \end{cases}$$

To set the threshold n, we consider
$$P_0(\operatorname{sgn}(Y) > n) = \begin{cases} 0 & \text{if } n \geq 1 \\ 1/2 & \text{if } -1 \leq n < 1 \\ 1 & \text{if } n < -1 \end{cases}$$

This implies that
$$n = \begin{cases} 1 & \text{if } 0 < \alpha < 1/2. \\ -1 & \text{if } 1/2 \leq \alpha < 1 \end{cases}$$

The randomization is.
$$\gamma = \frac{\alpha - P_0(\operatorname{sgn}(Y) > n)}{P_0(\operatorname{sgn}(Y) - n)} = \begin{cases} 2\alpha & \text{if } 0 < \alpha < 1/2 \\ 2\alpha - 1 & \text{if } 1/2 \leq \alpha < 1. \end{cases}$$

i.e. LMP test is thus.

$$\delta_{l_0}(y) = \begin{cases} 2\alpha & \text{if } y > 0 \\ 0 & \text{if } y \leq 0. \end{cases}$$

for $0 < \alpha < 1/2$; and it is.

$$\delta_{l_0}(y) = \begin{cases} 1 & \text{if } y \geq 0 \\ 2\alpha - 1 & \text{if } y < 0 \end{cases}$$

for $1/2 \leq \alpha < 1$

For fixed $\theta > 0$, detection probability is

$$P_D(\tilde{\delta}_{l_0}; \theta) = P_\theta(\mathrm{sgn}(Y) > \eta) + \gamma P_\theta(\mathrm{sgn}(Y) = \eta)$$

$$= \begin{cases} 2\alpha \int_0^\infty \frac{1}{2} e^{-|y-\theta|} dy & \text{if } 0 < \alpha < 1/2 \\ \int_0^\infty \frac{1}{2} e^{-|y-\theta|} dy + (2\alpha - 1) \int_{-\infty}^0 \frac{1}{2} e^{-|y-\theta|} dy & \text{if } 1/2 \leq \alpha < 1. \end{cases}$$

$$= \begin{cases} \alpha(2 - e^{-\theta}) & \text{if } 0 < \alpha < 1/2. \\ 1 + (\alpha - 1) e^{-\theta} & \text{if } 1/2 \leq \alpha < 1. \end{cases}$$

b.) For fixed θ, the NP critical region.

$$\Gamma_\theta = \{ |y| - |y-\theta| > \eta' \}.$$

$$= \begin{cases} (-\infty, \infty) & \text{if } \eta' < -\theta. \\ \left(\left(\frac{\eta'+\theta}{2}\right), \infty\right) & \text{if } -\theta \le \eta' \le \theta. \\ \phi & \text{if } \eta' > \theta. \end{cases}$$

from which.

$$P_d(\Gamma_\theta) = \begin{cases} 1 & \text{if } \eta' < -\theta. \\ \frac{1}{2} e^{-(\eta'+\theta)/2} & \text{if } -\theta \le \eta' \le \theta. \\ 0 & \text{if } \eta' > \theta \end{cases}$$

Clearly, we must know θ to set η', and thus the NP critical region depends on θ. This implies that there is no UMP test.

The generalized likelihood ratio test uses this statistic:

$$\sup_{\theta > 0} e^{|y| - |y-\theta|} = \exp\left\{ \sup_{\theta > 0} \left(|y| - |y-\theta| \right) \right\}.$$

$$= \begin{cases} 1 & \text{if } y < 0 \\ e^{y} & \text{if } y \ge 0, \end{cases}$$

4. $H_0 : \underline{Y} = \underline{N} + \underline{S_0}$

$H_1 : \underline{Y} = \underline{N} + \underline{S_1}$

$\vdots$

$H_{M-1} : \underline{Y} = \underline{N} + \underline{S_{M-1}}$

$\|s_0\|^2 = \|s_1\|^2 = \dots = \|S_{M-1}\|^2$.

(a). $$P_e = 1 - \left[\sum_{i=0}^{M-1} \int_{\Gamma_i} \frac{1}{(\sqrt{2\pi\sigma^2})^{n/2}} e^{-\frac{(y-s_i)^T(y-s_i)}{\sigma^2}} dy \right] \times \frac{1}{M}$$

Minimising P_e is equivalent to choosing

Γ_i such that $\bigcup_{i=0}^{M-1} \Gamma_i = R^n$, ~~such~~ that

results in maximum $\sum_{i=0}^{M-1} \int_{\Gamma_i} \frac{1}{(\sqrt{2\pi\sigma^2})^{n/2}} e^{-\frac{(y-s_i)^T(y-s_i)}{\sigma^2}} dy$

Claim: The above term is maximised by choosing the shortest Euclidean distance decoder.

Proof by contradiction.

Let. $\Gamma' \in \Gamma_0$ (without loss of generality) but

for any x in Γ' $(x-s_0)^T(x-s_0) > (x-s_1)^T(x-s_1)$

i.e Γ' is ~~closer~~ nearer to s_1 than s_0.

Thus, $$\int_{\Gamma'} \frac{1}{(\sqrt{2\pi\sigma^2})^{n/2}} e^{-\frac{(y-s_0)^T(y-s_0)}{\sigma^2}} dy < \int_{\Gamma'} \frac{1}{(\sqrt{2\pi\sigma^2})^{n/2}} e^{-\frac{(y-s_1)^T(y-s_1)}{\sigma^2}} dy$$

Thus, the probability of correct detection would be maximised if $\Gamma' \in \Gamma_i$ when Γ' is nearest to s_i.

5. $Y_k = N_k - s_k.$

$Y_k = N_k.$

$Y_k = N_k + s_k.$

$s = [s_1, s_2, \dots s_N]$ is a known signal seq.

$\underline{s}, \underline{o}, -\underline{s}$ are linear. So under some rotation the problem can be rewritten as

$H_0: y_K = \underline{N}' - [|s|, 0, 0, \dots]$

$H_1: y_K = \underline{N}'$

$H_2: y_K = \underline{N}' + [|s|, 0, 0, \dots]$

$\because$ $\underline{N}$ is uncorrelated 0-mean 1-variance.

$\underline{N}'$ can also be shown to be 0-mean, 1-variance.

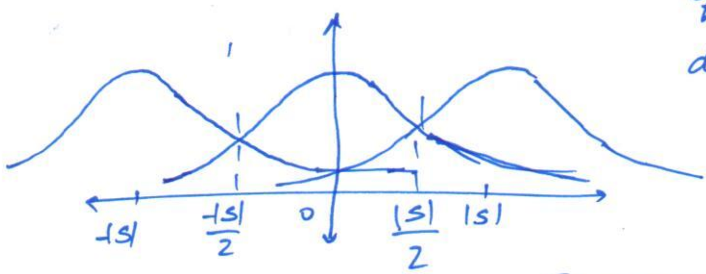
$\therefore$ This problem is simply reduced to a 1-D problem.

$H_0: \; y = n - |s|$

$H_1: \; y = n$

$H_2: \; y = n + |s|$

for which we know that min. probability of error is achieved by the min. euclidean distance ~~deto~~ detector.



$\cancel{\delta = \left\{ 0, \quad y < \frac{|s|}{2} \right.}$

For the original problem $\delta = \begin{cases} 0, & \frac{\underline{y}\cdot\underline{s}}{|\underline{s}|} < -\frac{|\underline{s}|}{2} \\ 2, & \frac{\underline{y}\cdot\underline{s}}{|\underline{s}|} > \frac{|\underline{s}|}{2} \\ 1, & \text{otherwise.} \end{cases}$

Min. $P_e = \frac{4}{3} Q\left(\frac{|\underline{s}|}{2}\right)$

5. (b). ~~$P_e = 1 - \frac{1}{\cdot}$~~

when all signals are orthogonal the signals maybe represented in a rotated space as.

$[\|S_0\|, 0, 0, \dots]$

$[0, \|S_0\|, 0, \dots]$

$[0, 0, \|S_0\|, 0 \dots]$

and so on.

$$P_e = 1 - \frac{1}{M} \sum_{i=0}^{M-1} \int_{\Gamma_i} \frac{e^{-\frac{(y-s_i)^T(y-s_i)}{2\sigma^2}}}{(\sqrt{2\pi\sigma^2})^{n/2}} \, dy.$$

$$= \frac{1}{} - \frac{1}{M} \sum_{i=0}^{M-1} \int_{\Gamma_i'} \frac{e^{-\frac{(\underline{x}-\underline{d_i})^T(\underline{x}-\underline{d_i})}{2}}}{(\sqrt{2\pi})^{n/2}} \, dx.$$

$$d_i = \frac{s_i}{\sigma} \qquad \underline{x} = \frac{y}{\sigma}$$

$\because$ From part(a) we know that this is the shortest distance detector.

$$\int_{\Gamma_i'} \frac{e^{-\frac{(\underline{x}-\underline{d}_i)^T(\underline{x}-\underline{d_i})}{2}}}{(\sqrt{2\pi})^n} \, d\underline{x} = \int_{-\infty}^{\infty} [\phi(x)]^{M-1} e^{-\frac{(x-d)^2}{2}} \, dx.$$

$$\text{where} \quad d = \frac{\|s_0\|}{\sigma}$$

$\because$ The region closest to s_i is given by $\{y : y_i \text{ is the highest component}\}$

$$y_k = N_k + \theta S_k \qquad k = 1, 2, \ldots n$$

$$E[N_k N_l] = \sigma^2 \rho^{|k-l|}$$

To show:

$$\delta(y) = \begin{cases} 1 & \sum_{k=1}^{n} b_k z_k \geq \tau' \\ 0 & \sum_{k=1}^{n} b_k z_k < \tau' \end{cases}$$

where $b_1 = \dfrac{s_1}{\sigma}$; $z_1 = \dfrac{y_1}{\sigma}$

$$b_k = \frac{(s_k - \rho s_{k-1})}{\sigma\sqrt{1-\rho^2}} \quad ; \quad z_k = \frac{(y_k - \rho y_{k-1})}{\sigma\sqrt{1-\rho^2}}$$

Sol:

$$\delta(y) = \begin{cases} 1 & \text{if } LLR \geq \tau \\ 0 & LLR < \tau \end{cases}$$

$$LLR = \frac{\dfrac{1}{\sqrt{(2\pi)^n |C|}} e^{-\frac{1}{2}(y-s)^T C^{-1}(y-s)}}{\dfrac{1}{\sqrt{(2\pi)^n |C|}} e^{-\frac{1}{2} y^T C^{-1} y}}$$

where

$$C = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & & \\ \vdots & \rho & & & \\ \vdots & & & 1 & \\ \rho^{n-1} & \cdots & \cdots & \rho & 1 \end{bmatrix}$$

$$= e^{-\frac{1}{2}(y-s)^T C^{-1}(y-s) + \frac{1}{2} y^T C^{-1} y}$$

$$= e^{+\frac{1}{2} s^T C^{-1} y + \frac{1}{2} y^T C^{-1} s - \frac{1}{2} s^T C^{-1} s}$$

$$= e^{s^T C^{-1} y - \frac{1}{2} s^T C^{-1} s}$$

$$LLR \geq \tau \Rightarrow \log(LLR) \geq \log \tau$$

$$\Rightarrow s^T C^{-1} y - \frac{1}{2} s^T C^{-1} s \geq \log \tau$$

$$\Rightarrow s^T C^{-1} y \geq \log \tau + \underbrace{\frac{1}{2} s^T C^{-1} s}_{\text{independant of } y}$$

$$\Rightarrow s^T C^{-1} y \geq \tau'$$

$$\delta(y) = \begin{cases} 1 & \text{if } s^T C^{-1} y \geq \tau' \\ 0 & \text{if } s^T C^{-1} y < \tau' \end{cases}$$

In the question we have.

$$\delta(y) = \begin{cases} 1 & \text{if } s^T B^T B y \geq \tau' \\ 0 & \text{if } s^T B^T B y < \tau' \end{cases}$$

where

$$B^T = \frac{1}{\sigma} \begin{bmatrix} 1 & \frac{-e}{\sqrt{1-e^2}} & 0 & & 0 \\ 0 & \frac{1}{\sqrt{1-e^2}} & \frac{-e}{\sqrt{1-e^2}} & \ddots & \\ 0 & 0 & \frac{1}{\sqrt{1-e^2}} & \ddots & \frac{-e}{\sqrt{1-e^2}} \\ & 0 & & & \frac{1}{\sqrt{1-e^2}} \end{bmatrix}$$

So we need to proove $B^T B = C^{-1}$ (or) $B^T B C = I$

$$B^T B = \frac{1}{\sigma^2(1-e^2)} \begin{bmatrix} \sqrt{1-e^2} & -e & 0 & \cdots \\ 0 & 1 & -e & \cdots \\ 0 & 0 & 1 & \ddots & -e \\ & 0 & & & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1-e^2} & 0 & & \\ -e & 1 & & 0 \\ 0 & -e & 1 & \\ & 0 & \ddots & \\ & & -e & 1 \end{bmatrix}$$

$$= \frac{1}{\sigma^2(1-e^2)} \begin{bmatrix} 1 & -e & 0 & \cdots & 0 \\ -e & 1+e^2 & -e & \cdots & 0 \\ 0 & -e & 1+e^2 & -e & \cdots 0 \\ & \ddots & \ddots & \ddots & -e \\ & 0 & & -e & 1+e^2 \end{bmatrix}$$

$$S'BC = \frac{1}{\sigma^2(1-e^2)} \begin{bmatrix} 1 & -e & 0 & & 0 \\ -e & 1+e^2 & -e & & \\ 0 & -e & 1+e^2 & \ddots & \\ & & \ddots & \ddots & -e \\ 0 & & & -e & 1+e^2 \end{bmatrix} \sigma^2 \begin{bmatrix} 1 & e & e^2 & \cdots & e^{n-1} \\ e & 1 & e & & \\ \vdots & e & \ddots & & e \\ e^{n-1} & e^{n-2} & & e & 1 \end{bmatrix}$$

$$= \frac{1}{1-e^2} \begin{bmatrix} 1-e^2 & e-e & e^2-e^2 & 0 \\ -e+e+e^3-e^3 & 1-e^2 & & 0 \\ 0 & 0 & \ddots & \\ 0 & & & 1-e^2 \end{bmatrix}$$

$$= \underline{I}$$

b) To find ROC

~~&y~~ Let $T = \sum_{k=1}^{n} b_k z_k$

$$z_1 = \frac{y_1}{\sigma} \quad ; \quad z_k = \frac{(y_k - e\, y_{k-1})}{\sigma\sqrt{1-e^2}}$$

$\cancel{E[z_1]}$

$$E[z_1 ; H_0] = 0 \quad ; \quad E[z_k ; H_0] = 0$$

$$E[z_1 ; H_1] = \frac{\theta}{\sigma} \quad ; \quad E[z_k ; H_0] = \frac{\theta(s_k - e\, s_{k-1})}{\sigma\sqrt{1-e^2}}$$

$$\mathrm{Var}(z_k ; H_i) = \frac{1}{\sigma^2(1-e^2)} E\left[(y_k - e\, y_{k-1})^2\right] = \frac{1}{\sigma^2(1-e^2)}\left[\sigma^2 + e^2\sigma^2 - 2e^2\sigma^2\right]$$

$$= 1$$

$$\cancel{\mathrm{var}}\; E[z_k z_j]_{j \neq k} = 0$$

$$E[T ; H_1] =$$

$$T \sim \begin{cases} N\left(0, \sum_{k=1}^{n} b_k^2\right) & \text{under } H_0 \\ N\left(\theta \sum_{k=1}^{n} b_k^2, \sum_{k=1}^{n} b_k^2\right) & \text{under } H_1 \end{cases}$$

$$P_{FA} = \mathbb{P}\{ T \geq \tau' ; H_0 \}$$

$$\alpha = Q\left(\frac{\tau'}{\sqrt{\sum_{k=1}^{n} b_k^2}}\right)$$

$$\tau' = \sqrt{\sum_{k=1}^{n} b_k^2}\; Q^{-1}(\alpha)$$

$$P_D = \mathbb{P}\{ T \geq \tau' ; H_1 \}$$

$$= Q\left(\frac{\tau' - \theta \sum b_k^2}{\sqrt{\sum b_k^2}}\right) = Q\left(\frac{\sqrt{\sum b_k^2}\, Q^{-1}(\alpha) - \theta \sum b_k^2}{\sqrt{\sum b_k^2}}\right)$$

$$= Q\left(Q^{-1}(\alpha) - \frac{\theta \sum b_k^2}{\sqrt{\sum b_k^2}}\right) = Q\left(Q^{-1}(\alpha) - \frac{\theta}{\sigma} \underbrace{\sqrt{s_1^2 + \sum_{k=2}^{n} \frac{(s_k - \rho s_{k-1})^2}{(1-\rho^2)}}}_{d^2}\right)$$

if $s_i = A \;\forall i$ then

$$P_D = Q\left(Q^{-1}(\alpha) - \frac{\theta}{\sigma}\sqrt{A^2\left(1 + \frac{(1-\rho)(n-1)}{1+\rho}\right)}\right)$$

→ As ρ increases d^2 decreases.

As n increases d^2 increases.