

Tutorial - 1

1. Refer Notes

$$2. p_0(y) = \begin{cases} (2/3)(y+1) & , 0 \leq y < 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$p_1(y) = \begin{cases} 1 & , 0 \leq y < 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$c_{ij} = \begin{cases} 0 & , i=j \\ 1 & , i \neq j \end{cases}$$

$$\Gamma_0 = \left\{ y \in [0, 1] \mid \frac{\frac{2}{3}(y+1)}{1} \geq \frac{\pi_1}{\pi_0} \right\}$$

$$\Gamma_0 = \left\{ y \geq \frac{1}{2} \text{ and } y < 1 \right\} \xrightarrow{\text{Bayes rule}} (\Gamma_1 = \Gamma_0^c)$$

$$\text{Min. Bayes Risk: } \frac{1}{2} \int_{\Gamma_1} p_0(y) dy + \frac{1}{2} \int_{\Gamma_0} p_1(y) dy$$

$$= \frac{1}{2} \left[1 + \int_{\Gamma_0} (p_1(y) - p_0(y)) dy \right]$$

$$= \frac{1}{2} \left[1 + \int_{1/2}^1 \left[1 - \frac{2}{3}(y+1) \right] dy \right]$$

$$= \frac{1}{2} \left[1 + \int_{1/2}^1 \left[\frac{1}{3} - \frac{2}{3}y \right] dy \right]$$

$$= \frac{1}{2} \left[1 + \frac{1}{3} \left[y - y^2 \right]_{1/2}^1 \right]$$

$$= \frac{1}{2} \left[1 + \frac{1}{3} \left[0 - \frac{1}{4} \right] \right]$$

$$= \frac{1}{2} \left[\frac{11}{12} \right] = \frac{11}{24} \xrightarrow{\text{Min. Bayes risk.}}$$

3.

$$p_j(y) = \frac{(j+1)}{2} e^{-(j+1)|y|}$$

$$c_{ij} = \begin{cases} 0, & i=j \\ 1, & i=1, j=0 \\ 3/4, & i=0, j=1 \end{cases}$$

$$p_1(y) = e^{-2|y|}$$

$$p_0(y) = \frac{1}{2} e^{-|y|}$$

$$\pi_0 = 1/4, \pi_1 = 3/4$$

$$r_1 = \left\{ \frac{p_1(y)}{p_0(y)} \geq \frac{\pi_0 (c_{10} - c_{00})}{\pi_1 (c_{01} - c_{11})} \right\}$$

$$= \left\{ \frac{(2/2) e^{-2|y|}}{(1/2) e^{-|y|}} \geq \frac{1}{3} \left(\frac{1}{3/4} \right) \right\}$$

$$= \left\{ 2e^{-|y|} \geq \frac{4}{9} \right\} = \left\{ -|y| \geq \ln\left(\frac{2}{9}\right) \right\}$$

$$= \left\{ y \in \left[-\ln\left(\frac{9}{2}\right), \ln\left(\frac{9}{2}\right) \right] \right\} \longrightarrow \text{Decision region for Bayes rule.}$$

$$\text{Min. Bayes Risk} = \int_{r_1} \pi_0 c_{10} p_0(y) dy + \int_{r_0} \pi_1 c_{01} p_1(y) dy$$

$$= \int_{r_1} [\pi_0 c_{10} p_0(y) - \pi_1 c_{01} p_1(y)] dy + \pi_1 c_{01}$$

$$= 2 \int_0^{\ln(9/2)} \left[\frac{1}{4} \times \frac{1}{2} e^{-y} - \frac{9}{16} e^{-2y} \right] dy + \frac{9}{16}$$

$$= 2 \int_0^{\ln(9/2)} \left(\frac{1}{8} e^{-y} - \frac{9}{16} e^{-2y} \right) dy + \frac{9}{16}$$

$$= 2 \left[\frac{9}{32} e^{-2y} - \frac{1}{8} e^{-y} \right]_0^{\ln(9/2)} + \frac{9}{16}$$

$$= 2 \left[\frac{9}{32} \times \frac{1}{9} - \frac{1}{8} \times \frac{2}{9} + \frac{1}{8} \right] + \frac{9}{16}$$

$$= 2 \left[\frac{1}{32} + \frac{1}{16} \right] + \frac{9}{16}$$

$$= \frac{6}{16} + \frac{9}{16} = \frac{15}{16}$$

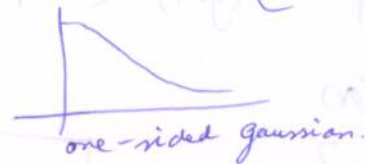
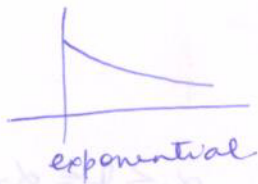
$$= 2 \left[\frac{1}{32} \times \frac{1}{9} - \frac{1}{32} - \frac{1}{84} \times \frac{2}{9} + \frac{1}{8} \right] + \frac{9}{16}$$

$$= \frac{1}{2} - \frac{1}{18} + \frac{9-9}{16} = \frac{8}{18} = \frac{4}{9}$$

④

$$p_0(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$p_1(y) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-y^2/2} & y \geq 0 \\ 0 & y < 0 \end{cases}$$



Priors: π_0, π_1 .

Uniform costs: $C_{00} = C_{11} = 0$; $C_{10} = C_{01} = 1$.

$$L(y) = \frac{p_1(y)}{p_0(y)} = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-y^2/2} \cdot e^y & y \geq 0 \\ \emptyset & \text{else. (not needed)} \end{cases}$$

$$L(y) > \frac{\pi_0}{\pi_1} \quad \text{Decide } H_1$$

$$L(y) < \frac{\pi_0}{\pi_1} \quad \text{Decide } H_0$$

$$L(y) = \frac{\pi_0}{\pi_1} \quad \text{Decide } H_0 \text{ or } H_1.$$

$$L(y) > \frac{\pi_0}{\pi_1} \iff \sqrt{\frac{2}{\pi}} e^{-y^2/2} e^y > \frac{\pi_0}{\pi_1}$$

$$y(1 - \frac{y}{2}) > \ln\left(\frac{\pi_0}{\pi_1} \sqrt{\frac{\pi}{2}}\right)$$

$$\frac{y^2}{2} - y + \ln\left(\frac{\pi_0}{\pi_1} \sqrt{\frac{\pi}{2}}\right) < 0$$

$$y^2 - 2y + 2 \ln\left(\frac{\pi_0}{\pi_1} \sqrt{\frac{\pi}{2}}\right) < 0$$

$$(y-1)^2 + 2 \ln\left(\frac{\pi_0}{\pi_1} \sqrt{\frac{\pi}{2}}\right) - 1 < 0$$

$$(y-1)^2 < 1 - 2 \ln\left(\frac{\pi_0}{\pi_1} \sqrt{\frac{\pi}{2}}\right)$$

Case 1

if $2 \ln\left(\frac{\pi_0}{\pi_1} \sqrt{\frac{\pi}{2}}\right) > 1$, this is not feasible \Rightarrow rule will be Decide H_0 always

$$(or) \left(\frac{\pi_0}{\pi_1}\right)^2 \frac{\pi}{2} > e \quad (or) \quad \frac{\pi_0}{\pi_1} > \sqrt{\frac{2e}{\pi}} \quad (or) \quad \boxed{\pi_0 > \frac{\sqrt{2e/\pi}}{1 + \sqrt{2e/\pi}}}$$

Case 2

if $2 \ln\left(\frac{\pi_0}{\pi_1} \sqrt{\frac{\pi}{2}}\right) < 1$

Decide H_1 if $(y-1)^2 < 1 - 2 \ln\left(\frac{\pi_0}{\pi_1} \sqrt{\frac{\pi}{2}}\right)$

(or)

$\max\left\{-\sqrt{1 - 2 \ln\left(\frac{\pi_0}{\pi_1} \sqrt{\frac{\pi}{2}}\right)} + 1, 0\right\} \leq y \leq \sqrt{1 - 2 \ln\left(\frac{\pi_0}{\pi_1} \sqrt{\frac{\pi}{2}}\right)} + 1$ $\alpha_1 \leq y \leq \alpha_2$

Minimum Bayes risk

Case 1 : Decide H_0 always

Minimum Bayes risk = π_1

Case 2 : $\pi_0 \int_{\alpha_1}^{\alpha_2} p_0(y) dy + \pi_1 \left[\int_0^{\alpha_1} p_1(y) dy + \int_{\alpha_2}^{\infty} p_1(y) dy \right]$

= $\pi_0 [e^{-\alpha_1} - e^{-\alpha_2}] + \pi_1 [2(Q(0) - Q(\alpha_1)) + 2(Q(\alpha_2))]$

= $\pi_0 [e^{-\alpha_1} - e^{-\alpha_2}] + \pi_1 [1 - 2Q(\alpha_1) + 2Q(\alpha_2)]$

$\left(\frac{\pi}{2}\right) \frac{dy}{\pi} < (y-1)^2$

$0 > \left(\frac{\pi}{2}\right) \frac{dy}{\pi} + (y-1)^2$

$0 > \left(\frac{\pi}{2}\right) \frac{dy}{\pi} + (y-1)^2$

$0 > 1 - \left(\frac{\pi}{2}\right) \frac{dy}{\pi} + (1-y)^2$

$\left(\frac{\pi}{2}\right) \frac{dy}{\pi} - 1 > (1-y)^2$

$\frac{\pi/2}{\pi} < 1$

(i) $\frac{dy}{\pi} < \frac{1}{\pi}$ (ii) $\frac{dy}{\pi} > \frac{1}{\pi}$ (iii) $\frac{dy}{\pi} < \frac{1}{\pi}$

Case 1