Tutorial - 1

1. Refer Notes

2. $$p_0(y) = \begin{cases} (2/3)(y+1) & , 0 \le y \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$p_1(y) = \begin{cases} 1 & , 0 \le y \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$c_{ij} = \begin{cases} 0 & , i = j \\ 1 & , i \ne j \end{cases}$$

$$\Gamma_0 = \left\{ y \in [0,1] \;\middle|\; \frac{\frac{2}{3}(y+1)}{1} \ge \frac{\pi_1}{\pi_0} \; 1 \right\}$$

$\Gamma_0 = \{ y \ge \frac{1}{2} \text{ and } y \le 1 \}$ → Bayes rule $(\Gamma_1 = \Gamma_0^c)$

Min. Bayes Risk: $$\frac{1}{2}\int_{\Gamma_1} p_0(y)\,dy + \frac{1}{2}\int_{\Gamma_0} p_1(y)\,dy$$

$$= \frac{1}{2}\left[1 + \int_{\Gamma_0} (p_1(y) - p_0(y))\,dy\right]$$

$$= \frac{1}{2}\left[1 + \int_{1/2}^{1} \left[1 - \frac{2}{3}(y+1)\right] dy\right]$$

$$= \frac{1}{2}\left[1 + \int_{1/2}^{1} \frac{1}{3} - \frac{2}{3}y \,dy\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{3}\left[y - y^2\right]_{1/2}^{1}\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{3}\left[0 - \frac{1}{4}\right]\right]$$

$$= \frac{1}{2}\left[\frac{11}{12}\right] = \frac{11}{24}$$ → Min. Bayes risk.

3. $p_j(y) = \left(\frac{j+1}{2}\right) e^{-(j+1)|y|}$

$p_1(y) = e^{-2|y|}$

$p_0(y) = \frac{1}{2} e^{-|y|}$

$$c_{ij} = \begin{cases} 0, & i=j \\ 1, & i=1,\ j=0 \\ 3/4, & i=0,\ j=1 \end{cases}$$

$\pi_0 = 1/4, \quad \pi_1 = 3/4$

$$\Gamma_1 = \left\{ y \ \frac{p_1(y)}{p_0(y)} \geq \frac{\pi_0}{\pi_1} \frac{(c_{10} - c_{00})}{(c_{01} - c_{11})} \right\}$$

$$= \left\{ \frac{(2/2) e^{-2|y|}}{(1/2) e^{-|y|}} \geq \frac{1}{3}\left(\frac{1}{3/4}\right) \right\}$$

$$= \left\{ \cancel{2} e^{-|y|} \geq \frac{\cancel{4}\, 2}{9} \right\} = \left\{ -|y| \geq \ln\left(\frac{2}{9}\right) \right\}$$

$$= \left\{ y \in \left[-\ln\left(\frac{9}{2}\right), \ln\left(\frac{9}{2}\right)\right] \right\} \longrightarrow \text{Decision region for Bayes rule.}$$

Min. Bayes Risk $= \int_{\Gamma_1} \pi_0 c_{10} p_0(y)\,dy + \int_{\Gamma_0} \pi_1 c_{01} p_1(y)\,dy$

$$= \int_{\Gamma_1} \left[\pi_0 c_{10} p_0(y) - \pi_1 c_{01} p_1(y)\right] dy + \pi_1 c_{01}$$

$$= 2\int_0^{\ln(9/2)} \left[\frac{1}{4} \times \frac{1}{2} e^{-y} - \frac{9}{16} e^{-2y}\right] dy + \frac{9}{16}$$

$$= 2\int_0^{\ln(9/2)} \left(\frac{1}{8} e^{-y} - \frac{9}{16} e^{-2y}\right) dy + \frac{9}{16}$$

$$= 2\left[\frac{9}{32} e^{-2y} - \frac{1}{8} e^{-y}\right]_0^{\ln(9/2)} + \frac{9}{16}$$

$$\cancel{= 2\left[\frac{9}{32} \times -9 - \frac{9}{32} - \frac{1}{8} \times \frac{-9}{2} + \frac{1}{8}\right] + \frac{9}{16}}$$

$$\cancel{= 2\left[\frac{45}{16} + \frac{11}{16}\right] + \frac{9}{16}}$$

$$\cancel{= \frac{-68}{16} + \frac{9}{16} = \frac{-59}{16}}$$

$$= 2\left[\frac{\cancel{9}}{\cancel{32}\,8} \times \frac{\cancel{4}}{\cancel{9}} - \frac{9}{32} - \frac{1}{\cancel{8}4} \times \frac{\cancel{2}}{9} + \frac{1}{8}\right] + \frac{9}{16}$$

$$= \frac{1}{2} - \frac{1}{18} + \cancel{\frac{9}{16}} - \cancel{\frac{9}{16}} = \frac{8}{18} = \frac{4}{9}$$

(4)

$$p_0(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases} \qquad p_1(y) = \begin{cases} \sqrt{\frac{2}{\pi}}\, e^{-y^2/2} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

exponential

one-sided gaussian.

Priors : π_0, π_1.

Uniform costs : $C_{00} = C_{11} = 0$; $C_{10} = C_{01} = 1$.

$$L(y) = \frac{p_1(y)}{p_0(y)} = \begin{cases} \sqrt{2/\pi}\, e^{-y^2/2} \cdot e^{y} & y \geq 0 . \\ \emptyset & \text{else (not needed)} \end{cases}$$

$L(y) > \frac{\pi_0}{\pi_1}$ Decide H_1

$L(y) < \frac{\pi_0}{\pi_1}$ Decide H_0

$L(y) = \frac{\pi_0}{\pi_1}$ Decide H_0 or H_1.

$L(y) > \frac{\pi_0}{\pi_1}$ if $\sqrt{\frac{2}{\pi}}\, e^{-y^2/2} e^{y} > \frac{\pi_0}{\pi_1}$

$$y\left(1 - \frac{y}{2}\right) > \ln\left(\frac{\pi_0}{\pi_1}\sqrt{\frac{\pi}{2}}\right)$$

$$\frac{y^2}{2} - y + \ln\left(\frac{\pi_0}{\pi_1}\sqrt{\frac{\pi}{2}}\right) < 0$$

$$y^2 - 2y + 2\ln\left(\frac{\pi_0}{\pi_1}\sqrt{\frac{\pi}{2}}\right) < 0$$

$$(y-1)^2 + 2\ln\left(\frac{\pi_0}{\pi_1}\sqrt{\frac{\pi}{2}}\right) - 1 < 0 .$$

$$(y-1)^2 < 1 - 2\ln\left(\frac{\pi_0}{\pi_1}\sqrt{\frac{\pi}{2}}\right) .$$

$\boxed{\text{Case 1}}$

if $2\ln\left(\frac{\pi_0}{\pi_1}\sqrt{\frac{\pi}{2}}\right) > 1$, this is not feasible $\Rightarrow$ rule will be Decide H_0 always

(or) $\left(\frac{\pi_0}{\pi_1}\right)^2 \frac{\pi}{2} > e$ (or) $\frac{\pi_0}{\pi_1} > \sqrt{\frac{2e}{\pi}}$ (or) $\boxed{\pi_0 > \frac{\sqrt{2e/\pi}}{1 + \sqrt{2e/\pi}}}$

$\boxed{\text{case 2}}$

if $2\ln\left(\frac{\pi_0}{\pi_1}\sqrt{\frac{\pi}{2}}\right) < 1$

Decide H_1 if $(y-1)^2 < 1 - 2\ln\left(\frac{\pi_0}{\pi_1}\sqrt{\frac{\pi}{2}}\right)$.

(or)

$$\max\left\{-\sqrt{1-2\ln\left(\frac{\pi_0}{\pi_1}\sqrt{\frac{\pi}{2}}\right)}+1, 0\right\} \le y \le \sqrt{1-2\ln\left(\frac{\pi_0}{\pi_1}\sqrt{\frac{\pi}{2}}\right)} + 1 \qquad \alpha_1 \le y \le \alpha_2$$

Minimum Bayes risk

Case 1 : Decide H_0 always

Minimum Bayes risk $= \pi_1$

Case 2 : $\pi_0 \int_{\alpha_1}^{\alpha_2} p_0(y)\,dy + \pi_1\left[\int_0^{\alpha_1} p_1(y)\,dy + \int_{\alpha_2}^{\infty} p_1(y)\,dy\right]$

$$= \pi_0\left[e^{-\alpha_1} - e^{-\alpha_2}\right] + \pi_1\left[2\left(Q(0) - Q(\alpha_1)\right) + 2\left(Q(\alpha_2)\right)\right]$$

$$= \pi_0\left[e^{-\alpha_1} - e^{-\alpha_2}\right] + \pi_1\left[1 - 2Q(\alpha_1) + 2Q(\alpha_2)\right].$$