Q 2 Solutions

1) $\mathcal{H}_0 : \underline{Y} = \underline{N}$

$\mathcal{H}_1 : \underline{Y} = \underline{N} + A\left[(1-\Theta)\underline{s}^{(0)} + \Theta\,\underline{s}^{(1)}\right]$

$\underline{N} \sim N(\underline{0}, I)$

$A > 0$

$\Theta \sim \text{unif}\{0,1\}$

3+4+3

$\underline{s}^{(0)}, \underline{s}^{(1)}$ orthonormal.

(a) A known. Find $L(y)$. Under $\mathcal{H}_1$, $\underline{Y} \sim N(A\underline{s}^{(0)}, I)$ or $N(A\underline{s}^{(1)}, I)$ w.p. $\frac{1}{2}, \frac{1}{2}$ resp

$$L(y) = \frac{p_1(y)}{p_0(y)} = \frac{\frac{1}{2}\cdot\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(y-A\underline{s}^{(0)})^T(y-A\underline{s}^{(0)})} + \frac{1}{2}\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(y-A\underline{s}^{(1)})^T(y-A\underline{s}^{(1)})}}{\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}y^Ty}}$$

$$= \frac{1}{2} e^{-\frac{1}{2}(A^2\underline{s}^{(0)T}\underline{s}^{(0)} - 2A\underline{s}^{(0)T}y)} + \frac{1}{2} e^{-\frac{1}{2}(A^2\underline{s}^{(1)T}\underline{s}^{(1)} - 2A\underline{s}^{(1)T}y)}$$

$$= \frac{1}{2} e^{-\left(\frac{A^2}{2} - A\underline{s}^{(0)T}y\right)} + \frac{1}{2} e^{-\left(\frac{A^2}{2} - A\underline{s}^{(1)T}y\right)}.$$

(b) $\mathcal{H}_0 : A = 0$

$\mathcal{H}_1 : A > 0.$

Find LMP.

$$\frac{\left.\frac{\partial p_1(y)}{\partial A}\right|_{A=0}}{p_0(y)} = \left.\frac{\partial}{\partial A} L(y)\right|_{A=0} = \left. -\left(A - \underline{s}^{(0)T}y\right)\frac{1}{2}e^{-\left(\frac{A^2}{2} - A\underline{s}^{(0)T}y\right)} - \left(A - \underline{s}^{(1)T}y\right)\frac{1}{2}e^{-\left(\frac{A^2}{2} - A\underline{s}^{(1)T}y\right)}\right|_{A=0}$$

(since $p_0(y)$ does not depend on A)

$$= \frac{1}{2}\left(+\underline{s}^{(0)T}y + \underline{s}^{(1)T}y\right)$$

$$= \frac{1}{2}\sum_{k=1}^{n} y_k\left(s_k^{(0)} + s_k^{(1)}\right)$$

$$\text{LMP:}\quad \delta(y) = \begin{cases} 1 & \\ \gamma & \sum_{k=1}^{n} y_k\left(s_k^{(0)} + s_k^{(1)}\right) \begin{matrix} > \\ = \\ < \end{matrix} \eta \\ 0 & \end{cases}$$

(c)

Under $\mathcal{H}_0$ $\underline{y} \sim N(\underline{0}, I)$

$$T(\underline{y}) = (\underline{s}^{(0)} + \underline{s}^{(1)})^T \underline{y} \sim N(\underline{0}, (\underline{s}^{(0)}+\underline{s}^{(1)})^T I (\underline{s}^{(0)}+\underline{s}^{(1)}))$$

$$N(\underline{0}, 2)$$

Randomization not needed $\Rightarrow \gamma = 0$ (or choose $\gamma = 1$).

Set $P_F = P_0[T(\underline{y}) > \eta] = \alpha$

$$Q\left(\frac{\eta}{\sqrt{2}}\right) = \alpha$$

$$\Rightarrow \boxed{\eta = \sqrt{2}\, Q^{-1}(\alpha)}$$

$$P_D = P_1[T(\underline{y}) > \eta]$$

Under $\mathcal{H}_1$, $\underline{y} \sim N(A\underline{s}^{(0)}, I)$ w.p. $\frac{1}{2}$

$N(A\underline{s}^{(1)}, I)$ w.p $\frac{1}{2}$

$$T(\underline{y}) \sim N(A, 2)$$

$$P_D(\text{for a given } A) = P_1[T(\underline{y}) > \eta]$$

$$= Q\left(\frac{\eta - A}{\sqrt{2}}\right) = Q\left(Q^{-1}(\alpha) - \frac{A}{\sqrt{2}}\right)$$

(2) $\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \sim N(\underline{0}, \sigma^2 I)$

$\mathcal{H}_0$: $\sigma^2 = 1$

$\mathcal{H}_1$: $\sigma^2 = 2$

$\pi_0 = \pi_1 = \frac{1}{2}$.

$$L(\underline{y}) = \frac{\frac{1}{(2\pi)^{n/2}|\Sigma_1|^{1/2}} e^{-\frac{1}{2}\underline{y}^T\Sigma_1^{-1}\underline{y}}}{\frac{1}{(2\pi)^{n/2}|\Sigma_0|^{1/2}} e^{-\frac{1}{2}\underline{y}^T\Sigma_0^{-1}\underline{y}}} = \frac{|\Sigma_0|^{1/2}}{|\Sigma_1|^{1/2}} e^{\frac{1}{2}\left(\underline{y}^T(\Sigma_0^{-1} - \Sigma_1^{-1})\underline{y}\right)}$$

where $\Sigma_0 = I$

$\Sigma_1 = 2I$.

$$\text{Optimal detector } \delta(\underline{y}) = \begin{cases} 1 & L(\underline{y}) > \tau = 1 \\ 0 & \leq \end{cases}$$

$|\Sigma_0| = 1$

$|\Sigma_1| = 4$

$$L(y) = \frac{1}{2} e^{\frac{1}{2} y^T (1 - \frac{1}{2}) y} = \frac{1}{2} e^{\frac{y^T y}{4}} > 1$$

$$\frac{y^T y}{4} > \ln 2$$

$$y^T y > 4 \ln 2$$

if $\boxed{\frac{y_1^2 + y_2^2}{2} > 2 \ln 2}$, decide H_1,

else H_0.

3 $\alpha = P_0 \left(\lambda_N (Y_1, \dots, Y_N) \geq B \right)$

3+2

$$\Gamma_1 = \{ y \in \mathbb{R}^\infty \mid \lambda_N (y_1, \dots, y_N) \geq B \}$$

$$= \bigcup_{n=1}^{\infty} Q_n \quad \text{where } Q_n = \{ y \in \mathbb{R}^\infty \mid N = n,\ \lambda_n (y_1, \dots, y_n) \geq B \}.$$

$$\alpha = \sum_{n=1}^{\infty} \int_{Q_n} \prod_{k=1}^{n} p_0(y_k) \mu(dy_k) \leq \sum_{n=1}^{\infty} B^{-1} \int_{Q_n} \prod_{k=1}^{n} p_1(y_k) \mu(dy_k)$$

$$= \frac{1}{B} P_1 \left(\lambda_N (Y_1, \dots, Y_N) \geq B \right)$$

$$= \frac{1}{B} \beta$$

$$\alpha \leq \frac{\beta}{B} \Rightarrow \boxed{B \leq \frac{\beta}{\alpha}.}$$

Similarly $1 - \beta = P_1 \left(\lambda_N (Y_1, \dots, Y_N) \leq A \right) \leq A P_0 \left(\lambda_N (Y_1, \dots, Y_N) \leq A \right)$

$$= A (1 - \alpha)$$

$$\Rightarrow \boxed{A \geq \frac{1 - \beta}{1 - \alpha}}$$