EE 511 Solutions to Problem Set 5

1. (a) The sample functions are shown in Figure 1.

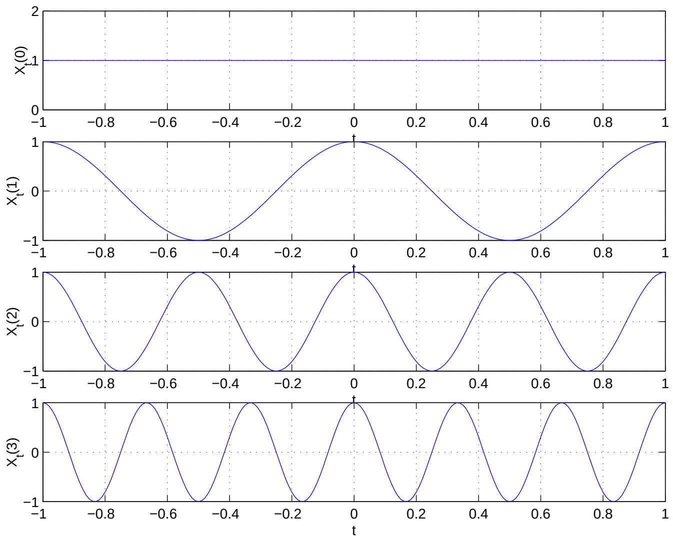

Figure 1:

(b) $X_0 = 0$. $X_{0.5}(0) = 1$, $X_{0.5}(1) = -1$, $X_{0.5}(2) = 1$, and $X_{0.5}(3) = -1$. $X_{0.25}(0) = 1$, $X_{0.25}(1) = 0$, $X_{0.25}(2) = -1$, and $X_{0.25}(3) = 0$. The marginal CDF's of X_0, $X_{0.25}$, and $X_{0.5}$ are shown in Figure 2.

(c) Given that $X_{0.5} = -1$, $X_{0.25} = 0$ with probability 1.

(d) Given that $X_{0.5} = 1$, $X_{0.25} = 1$ with probability 0.5 and $X_{0.25} = -1$ with probability 0.5.

2. a) $E[X_t] = 0$.

$$\begin{aligned} E[X_{t+\tau}X_t] &= E[A^2 \cos(2\pi f_c t + \Theta) \cos(2\pi f_c(t+\tau) + \Theta)] \\ &= \frac{A^2}{2}[\cos 2\pi f_c \tau + E[\cos(2\pi f_c(2t+\tau) + 2\Theta)]] \\ &= \frac{A^2}{2} \cos 2\pi f_c \tau \end{aligned}$$

b) We can choose any pdf for Θ as long as $E[\cos(2\pi f_c(2t+\tau) + 2\Theta)] = 0$ and $E[\cos(2\pi f_c t + \Theta)] =$ constant for any t, τ. Θ can be defined as follows:

$$\Theta = \begin{cases} 0 & \text{with prob.} \frac{1}{4} \\ \frac{\pi}{2} & \text{with prob.} \frac{1}{4} \\ \pi & \text{with prob.} \frac{1}{4} \\ \frac{3\pi}{2} & \text{with prob.} \frac{1}{4} \end{cases}$$

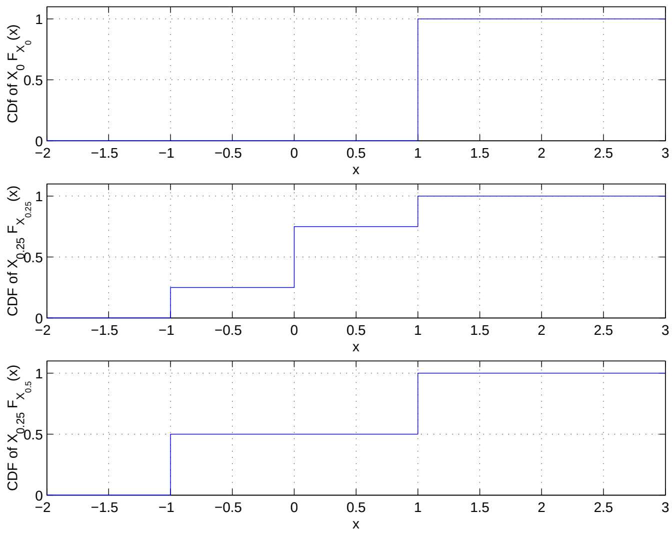

Figure 2:

Another possible choice for Θ is:

$$\Theta = \begin{cases} 0 & \text{with prob.} \frac{1}{12} \\ \frac{\pi}{4} & \text{with prob.} \frac{1}{6} \\ \frac{\pi}{2} & \text{with prob.} \frac{1}{12} \\ \frac{3\pi}{4} & \text{with prob.} \frac{1}{6} \\ \pi & \text{with prob.} \frac{1}{12} \\ \frac{5\pi}{4} & \text{with prob.} \frac{1}{6} \\ \frac{3\pi}{2} & \text{with prob.} \frac{1}{12} \\ \frac{7\pi}{4} & \text{with prob.} \frac{1}{6} \end{cases}$$

A more general choice for $f_\Theta(\theta)$ can be made as follows:

(i) Let us assume that the range of Θ is from 0 to 2π.

(ii) The condition for mean to be constant can be obtained as follows:

$$\begin{aligned} E[\cos(2\pi f_c t + \Theta)] &= \int_0^{2\pi} \cos(2\pi f_c t + \theta) f_\Theta(\theta) d\theta \\ &= \int_0^{\pi} \cos(2\pi f_c t + \theta) f_\Theta(\theta) d\theta + \int_\pi^{2\pi} \cos(2\pi f_c t + \theta) f_\Theta(\theta) d\theta \\ (\text{using } \theta' = \theta - \pi) &= \int_0^{\pi} \cos(2\pi f_c t + \theta) f_\Theta(\theta) d\theta + \int_0^{\pi} \cos(2\pi f_c t + \theta' + \pi) f_\Theta(\theta' + \pi) d\theta' \\ &= \int_0^{\pi} \cos(2\pi f_c t + \theta) f_\Theta(\theta) d\theta + \int_0^{\pi} [-\cos(2\pi f_c t + \theta')] f_\Theta(\theta' + \pi) d\theta' \\ &= \int_0^{\pi} \cos(2\pi f_c t + \theta)[f_\Theta(\theta) - f_\Theta(\theta + \pi)] d\theta \end{aligned}$$

Therefore, if $f_\Theta(\theta) = f_\Theta(\theta + \pi)$ for θ in $[0, \pi]$, then $E[\cos(2\pi f_c t + \Theta)] = 0$.

(iii) The additional condition for the auto-correlation funtion to be a function of τ can be obtained as follows:

$$\begin{aligned}
E[\cos(2\pi f_c(2t+\tau)+2\Theta)] &= \textstyle\int_0^{2\pi} \cos(2\pi f_c(2t+\tau)+2\theta) f_\Theta(\theta) d\theta \\
(\text{using } \phi = 2\theta) &= \tfrac{1}{2}\textstyle\int_0^{4\pi} \cos(2\pi f_c(2t+\tau)+\phi) f_\Theta\left(\tfrac{\phi}{2}\right) d\phi \\
&= \tfrac{1}{2}\textstyle\int_0^{2\pi} \cos(2\pi f_c(2t+\tau)+\phi) f_\Theta\left(\tfrac{\phi}{2}\right) d\phi \\
&\quad + \tfrac{1}{2}\textstyle\int_{2\pi}^{4\pi} \cos(2\pi f_c(2t+\tau)+\phi) f_\Theta\left(\tfrac{\phi}{2}\right) d\phi \\
(\text{using } \phi' = \phi - 2\pi) &= \tfrac{1}{2}\textstyle\int_0^{2\pi} \cos(2\pi f_c(2t+\tau)+\phi) f_\Theta\left(\tfrac{\phi}{2}\right) d\phi \\
&\quad + \tfrac{1}{2}\textstyle\int_0^{2\pi} \cos(2\pi f_c(2t+\tau)+\phi') f_\Theta\left(\tfrac{\phi'}{2}+\pi\right) d\phi' \\
&= \tfrac{1}{2}\textstyle\int_0^{2\pi} \cos(2\pi f_c(2t+\tau)+\phi) \left[f_\Theta\left(\tfrac{\phi}{2}\right) + f_\Theta\left(\tfrac{\phi'}{2}+\pi\right)\right] d\phi.
\end{aligned}$$

Assuming that we satisfy the condition from (ii) above, we get

$$E[\cos(2\pi f_c(2t+\tau)+2\Theta)] = \int_0^{2\pi} \cos(2\pi f_c(2t+\tau)+\phi) f_\Theta\left(\frac{\phi}{2}\right) d\phi.$$

Now, proceeding as in (ii), we need

$$f_\Theta\left(\frac{\phi}{2}\right) = f_\Theta\left(\frac{\phi+\pi}{2}\right)$$

for $\phi/2$ in $[0,\pi]$. Equivalently, we need

$$f_\Theta(\theta) = f_\Theta\left(\theta + \frac{\pi}{2}\right)$$

for θ in $[0,\pi/2]$.

(iv) Combining the conditions from (ii) and (iii), we get

$$f_\Theta(\theta) = f_\Theta\left(\theta + \frac{k\pi}{2}\right) \tag{1}$$

for θ in $[0,\pi/2]$ and $k = 1,2,3$. Therefore, we can choose any arbitrary $f_\Theta(\theta)$ for θ in $[0,\pi/2]$ such that

$$\int_0^{\frac{\pi}{2}} f_\Theta(\theta) d\theta = \frac{1}{4}.$$

$f_\Theta(\theta)$ for θ in $[\pi/2, 2\pi]$ can be set using (1).

A sample pdf that fives a W.S.S. X_t is shown in Figure 3.

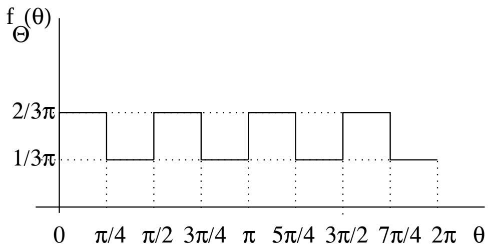

Figure 3:

3. a) $E[Y_t] = E[X_t \cos(2\pi f_c t + \Theta)]$. Since Θ and X_t are independent, $E[Y_t] = E[X_t]E[\cos(2\pi f_c t + \Theta$ $m_X E[\cos(2\pi f_c t + \Theta)] = 0$.

$$\begin{aligned} R_Y(t+\tau, t) &= E[X_t X_{t+\tau}]E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c(t+\tau) + \Theta)] \\ &= \tfrac{1}{2} R_X(\tau)[\cos 2\pi f_c \tau + E[\cos 2\pi f_c(2t+\tau) + 2\Theta]] \\ &= \tfrac{1}{2} R_X(\tau) \cos 2\pi f_c \tau. \end{aligned}$$

Y_t is W.S.S..

b) $E[Y_t] = E[X_t]\cos 2\pi f_c t = m_X \cos 2\pi f_c t$ is a function of time. Y_t is not W.S.S.

4. $E[X_t] = E[X_1]\cos 2\pi f_c t + E[X_2]\sin 2\pi f_c t$. For the mean to be independent of t, we need

$$E[X_1] = E[X_2] = 0.$$

$$\begin{aligned} R_X(t, t+\tau) &= E[(X_1 \cos 2\pi f_c(t+\tau) + X_2 \sin 2\pi f_c(t+\tau))(X_1 \cos 2\pi f_c t + X_2 \sin 2\pi f_c t)] \\ &= \left(\frac{E[X_1^2] + E[X_2^2]}{2}\right) \cos 2\pi f_c \tau \\ &+ 2E[X_1 X_2] \sin 2\pi f_c(2t+\tau) \\ &+ \left(\frac{E[X_1^2] - E[X_2^2]}{2}\right) \cos 2\pi f_c(2t+\tau). \end{aligned}$$

For $R_X(t, t+\tau)$ to be independent of t, we need

$$E[X_1 X_2] = 0 \quad \text{and} \quad E[X_1^2] = E[X_2^2].$$

The conditions derived above are both necessary and sufficient.

5.

$$
\begin{aligned}
E\left[|X_{t+\tau}-X_t|^2\right] &= E\left[X_{t+\tau}X_{t+\tau}^*\right]-E\left[X_{t+\tau}X_t^*\right]-E\left[X_tX_{t+\tau}^*\right]+E\left[X_tX_t^*\right] \\
\text{(Since } X_t \text{ is W. S. S.)} &= R_X(0)-R_X(\tau)-R_X^*(\tau)+R_X(0) \\
&= 2R_X(0)-2\text{Re}(R_X(\tau)) \\
\text{(Since } R_X(0) \text{ is real)} &= 2\text{Re}(R_X(0)-R_X(\tau)).
\end{aligned}
$$

6. (a)

$$
\begin{aligned}
X_0 &= 0 \\
\rho^{n-1}(X_1 &= W_1) \\
\rho^{n-2}(X_2 &= \rho X_1 + W_2) \\
\vdots \quad & \quad \vdots \\
\rho^0(X_n &= \rho X_{n-1}+W_n).
\end{aligned}
$$

Adding the above equations, we get

$$X_n = W_n + \rho W_{n-1} + \cdots + \rho^{n-1}W_1.$$

Therefore, $E[X_n]=0$ and $Var(X_n)=1+\rho^2+\cdots+\rho^{2n-2}$.

(b) $E[X_nX_{n+k}] = E[X_n(\rho X_{n+k-1}+W_{n+k})] = \rho E[X_nX_{n+k-1}]$. Therefore, we have $E[X_nX_{n+k}] = \rho^k E[X_n^2] = \rho^k(1+\rho^2+\cdots+\rho^{2n-2})$.

(c) No. $E[X_nX_{n+k}]$ is dependent on n.

7. Using Cauchy-Schwartz inequality and (geometric mean $\le$ arithmetic mean), we have

$$|R_{XY}(\tau)| \le \sqrt{R_X(0)R_Y(0)} \le 0.5[R_X(0)+R_Y(0)].$$

8. $R_X(t+\tau,t)=E[X_{t+\tau}X_t]=E[Y_{t+\tau}Z_{t+\tau}Y_tZ_t]$. Since Y_t and Z_t are independent random processes, $R_X(t+\tau,t)=E[Y_{t+\tau}Y_t]E[Z_{t+\tau}Z_t]=R_Y(\tau)R_Z(\tau)$. X_t is also W.S.S..

9. a) The transfer function of the filter (whose input is X_t and output is Y_t) is

$$H(f)=1-e^{-j2\pi fT}=1-\cos 2\pi fT + j\sin 2\pi fT.$$

$$
\begin{aligned}
S_Y(f) &= S_X(f)|H(f)|^2 \\
&= S_X(f)\left[(1-\cos 2\pi fT)^2+(\sin 2\pi fT)^2\right] \\
&= 2S_X(f)[1-\cos 2\pi fT)] = 4S_X(f)(\sin \pi fT)^2
\end{aligned}
$$

b) If $f \ll 1/T$ such that πfT is very small, then $\sin \pi fT$ is approximately equal to πfT. Therefore, $S_Y(f)=4\pi^2f^2T^2S_X(f)$. A scaled version of the same power spectral density would be obtained if Y_t is obtained from X_t using a differentiator, i.e., we will get $S_Y(f)=4\pi^2f^2S_X(f)$.

10. a) $E[Z_t] = E[X_t] + E[Y_t] = m_X + m_Y$.

$$\begin{aligned} R_Z(t,s) &= E[(X_t+Y_t)(X_s+Y_s)] \\ &= R_X(t,s) + R_{XY}(t,s) + R_{YX}(t,s) + R_Y(t,s) \\ \text{(using } \tau = t-s) &= R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau) \end{aligned}$$

Z_t is W.S.S..

b) $S_Z(f) = S_X(f) + S_{XY}(f) + S_{YX}(f) + S_Y(f)$.

c) If X_t and Y_t are uncorrelated and zero-mean, then $S_Z(f) = S_X(f) + S_Y(f)$. If they are non-zero mean random processes and uncorrelated, then $S_Z(f) = S_X(f) + S_Y(f) + 2m_X m_Y \delta(f)$.

11. (a) $R_S(t,s) = E[S_t S_s] = E[(X_t+Y_t)(X_s+Y_s)] = R_X(t,s) + R_{XY}(t,s) + R_{YX}(t,s) + R_Y(t,s)$. Since, X_t and Y_t are jointly W. S. S., we have

$$R_S(\tau) = R_X(\tau) + R_{XY}(\tau) + R_{XY}(-\tau) + R_Y(\tau).$$

Similarly, we can show

$$R_D(\tau) = R_X(\tau) - R_{XY}(\tau) - R_{XY}(-\tau) + R_Y(\tau).$$

(b) $R_{XS}(t,s) = E[X_t(X_s+Y_s)] = R_X(t,s) + R_{XY}(t,s)$. Therefore, we have $R_{XS}(\tau) = R_X(\tau) + R_{XY}(\tau)$.

(c) $R_{SD}(t,s) = E[(X_t+Y_t)(X_s-Y_s)] = R_X(t,s) - R_{XY}(t,s) + R_{YX}(t,s) - R_Y(t,s)$. Therefore, we have $R_{SD}(\tau) = R_X(\tau) - R_{XY}(\tau) + R_{XY}(-\tau) - R_Y(\tau)$.

12.

$$\begin{aligned} R_{ZW}(t,s) &= E[Z_t W_s] \\ &= E\left[\int_{-\infty}^{\infty} h_1(\tau_1) X_{t-\tau_1} d\tau_1 \int_{-\infty}^{\infty} h_2(\tau_2) Y_{s-\tau_2} d\tau_2\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_2) E[X_{t-\tau_1} Y_{s-\tau_2}] d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_2) R_{XY}(t-s-\tau_1+\tau_2) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} h_1(\tau_1) \left[\int_{-\infty}^{\infty} h_2(\tau_2) R_{XY}(t-s-\tau_1+\tau_2) d\tau_2\right] d\tau_1 \end{aligned}$$

From the above result, we see that $R_{ZW}(t,s)$ is a function of $\tau = t-s$ and is the convolution of $R_{XY}(\tau)$, $h_1(\tau)$ and $h_2(-\tau)$. Therefore, we have

$$S_{ZW}(f) = S_{XY}(f) H_1(f) H_2^*(f).$$

13. (a) $Y_n = X_n + X_{n-1} + X_{n-2}$.

$$\phi_{X_n}(s) = E[e^{sX_n}] = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} e^{sk} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda e^s}{k!} = e^{-\lambda} e^{\lambda e^s} = e^{-\lambda(1-e^s)}.$$

Since X_n, X_{n-1}, and X_{n-2} are independent and identically distributed, we have

$$\phi_{Y_n}(s) = E[e^{sY_n}] = E[e^{sX_n}]E[e^{sX_{n-1}}]E[e^{sX_{n-2}}] = E[e^{sX_n}]^3 = e^{-3\lambda(1-e^s)}.$$

Therefore, Y_n is a Poisson random variable with parameter 3λ, i. e.,

$$P[Y_n = k] = e^{-3\lambda} \frac{(3\lambda)^k}{k!} \quad \forall k \geq 0.$$

(b)

$$\phi_{Y_n}(s) = E[e^{sY_n}] = E[e^{sX_n}]E[e^{sX_{n-1}}]E[e^{sX_{n-2}}] = e^{-(\lambda_n + \lambda_{n-1} + \lambda_{n-2})(1-e^s)}.$$

Therefore, Y_n is a Poisson random variable with parameter $\lambda_n + \lambda_{n-1} + \lambda_{n-2}$, i. e.,

$$P[Y_n = k] = e^{-(\lambda_n + \lambda_{n-1} + \lambda_{n-2})} \frac{(\lambda_n + \lambda_{n-1} + \lambda_{n-2})^k}{k!} \quad \forall k \geq 0.$$