EE 511 Solutions to Problem Set 1

1. (i) $A + \overline{A} = S$ and $A\overline{A} = \phi$. Therefore, $P(A) + P(\overline{A}) = P(S) = 1$ and $P(\overline{A}) = 1 - P(A)$.

   (ii) $P(\overline{A}) \geq 0$. Therefore, $P(A) \leq 1$.

   (iii) $\phi + S = S$ and $\phi S = \phi$. Therefore, $P(\phi) + P(S) = P(S)$ and $P(\phi) = 0$.

   (iv) $B = BS = B(A_1 + ... + A_n)$. Since BA_i and BA_j are disjoint for $i \neq j$, $P(B) = P(BA_1) + P(BA_2) + ... + P(BA_n)$.

2. $B = A + \overline{A}B$ where $A(\overline{A}B) = \phi$. Therefore, $P(A) + P(\overline{A}B) = P(B)$. Since $P(\overline{A}B) \geq 0$, $P(A) \leq P(B)$.

3. $A + B = A + \overline{A}B$ with $A(\overline{A}B) = \phi$. Therefore, $P(A + B) = P(A) + P(\overline{A}B)$. Similarly, $B = (A + \overline{A})B = AB + \overline{A}B$. Therefore, $P(B) = P(AB) + P(\overline{A}B)$. Substituting this in the equation for $P(A + B)$, we get

$$P(A + B) = P(A) + P(B) - P(AB)$$

   Now,

$$\begin{aligned}
\text{P(A+B+C)} = \text{P((A+B)+C)} &= \text{P(A+B) + P(C) - P((A+B)C)} \\
&= \text{P(A) + P(B) - P(AB) + P(C) - P(AC + BC)} \\
&= \text{P(A) + P(B) + P(C) - P(AB) - (P(AC) + P(BC) - P(ACBC))} \\
&= \text{P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)}
\end{aligned}$$

4. We want to show $P(\sum_{i=1}^{N} A_i) \leq \sum_{i=1}^{N} P(A_i)$. This can be done in several ways.

   Solution 1:

   We have shown in problem 3 that $P(A + B) = P(A) + P(B) - P(AB)$, i.e., $P(A + B) \leq P(A) + P(B)$. Using this result repeatedly, we get

$$\begin{aligned}
P(\sum_{i=1}^{N} A_i) = P(A_1 + \sum_{i=2}^{N} A_i) &\leq P(A_1) + P(\sum_{i=2}^{N} A_i) \\
P(\sum_{i=2}^{N} A_i) = P(A_2 + \sum_{i=3}^{N} A_i) &\leq P(A_2) + P(\sum_{i=3}^{N} A_i) \\
\vdots \qquad &\vdots \qquad \vdots \\
P(\sum_{i=N-1}^{N} A_i) = P(A_{N-1} + A_N) &\leq P(A_{N-1}) + P(A_N)
\end{aligned}$$

   Combining the above equations, we get the desired result.

   Solution 2:

   We can write $\sum_{i=1}^{N} A_i$ as the sum of disjoint events $\sum_{i=1}^{N} B_i$ where $B_i = \overline{A_1}\ \overline{A_2}\ \cdots\ \overline{A_{i-1}}\ A_i$.

Now, for every i, we have $B_i \subset A_i$ and hence, using the result from problem 2, we have $P(B_i) \leq P(A_i)$. Therefore, we have

$$P(\sum_{i=1}^{N} A_i) = P(\sum_{i=1}^{N} B_i) = \sum_{i=1}^{N} P(B_i) \leq \sum_{i=1}^{N} P(A_i).$$

5. $P(AB) = P(A) + P(B) - P(A+B)$. Using $P(A+B) \leq 1$, $P(A) \geq 1-\delta$ and $P(B) \geq 1-\delta$, we get $P(AB) \geq 1-\delta+1-\delta-1$. Therefore, $P(AB) \geq 1-2\delta$.

6. $AB = A$. $P(A|B) = P(A)/P(B) = 3/4$. $P(B|A) = 1$.

7.

$$P(AB|C) = \frac{P(ABC)}{P(C)} = \frac{P(A|BC)P(BC)}{P(C)} = P(A|BC)P(B|C).$$

$$P(ABC) = P(AB|C)P(C) = P(A|BC)P(B|C)P(C)$$

8. We know, $P(A) > P(B) > P(C) > 0$, $A+B = S$, $AB = \phi$ and $P(AC) = P(A)P(C)$. We want to know if B and C can be disjoint. Let us evalaute $P(BC)$. If $BC = \phi$, $P(BC)$ should be 0.

   Since A and B partition S, we have $C = SC = (A+B)C = AC + BC$ and $P(C) = P(AC) + P(BC)$. Since A and C are independent, we have

$$P(C) = P(A)P(C) + P(BC)$$

   Therefore, we get

$$P(BC) = P(C)(1 - P(A))$$

   Since $A + B = S$ and $AB = \phi$, $P(A) + P(B) = P(S) = 1$. Therefore, $1 - P(A) = P(B)$. Using this, we get

$$P(BC) = P(C)P(B) > 0$$

   as $P(B) > 0$ and $P(C) > 0$. Since $P(BC) > 0$, B and C cannot be disjoint.

9. (i) $B = SB = (A + \overline{A})B = AB + \overline{A}B$. Using this, we get $P(B) = P(AB) + P(\overline{A}B)$. Now,

$$P(\overline{A}B) = P(B) - P(AB) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(\overline{A})P(B)$$

   Therefore, $\overline{A}$ and B are independent if A and B are independent.

   (ii) From (i), we know that given two independent events, complementing one of the events still gives two independent events. Therefore, if $\overline{A}$ and B are independent, $\overline{A}$ and $\overline{B}$ are independent. Since $\overline{A}$ and B are independent if A and B are independent, $\overline{A}$ and $\overline{B}$ are independent if A and B are independent.

   In fact, the following general result can be shown easily using the same technique used in part (i): If the events A_1, A_2, ..., A_n are independent and B_i equals A_i or $\overline{A_i}$ or S, then the events B_1, B_2, ..., B_n are also independent.

10. $P(A(B+C)) = P(AB+AC) = P(AB)+P(AC)-P(ABC)$. Since A, B, and C are independent, $P(A(B+C)) = P(A)P(B)+P(A)P(C)-P(A)P(B)P(C) = P(A)\left[P(B) + P(C) - P(BC)\right] = P(A)P(B+C)$. Thus, A and $B+C$ are independent.