EE 511 Problem Set 3

1. The receiver of a communication system receives random variable Y which is defined as $Y = X + N$ in terms of the input random variable X and the channel noise N. X takes on the values $-1/4$ and $1/4$ with $P[X = 1/4] = 0.6$. Let $f_N(n)$ denote the pdf of the channel noise and let X and N be independent. The receiver must decide for each received $Y = y$ whether the transmitted X was $-1/4$ or $1/4$. If N is uniform in $(-1/2, 1/2)$, (a) determine $f_Y(y|X = 1/4)$, $f_Y(y|X = -1/4)$ and $f_Y(y)$ (b) determine the optimal rule such that the probability of correct decision is maximised.

2. If random variables X and Y are related via $Y = g(X)$ where $g(.)$ is a monotonically increasing function, show that their CDF's satisfy $F_Y[g(\alpha)] = F_X(\alpha)$.

3. Let X be a random variable with pdf $f_X(x)$. Define $Y = g(X)$ where

$$g(x) = \begin{cases} x & |x| \leq 2 \\ -2 & x < -2 \\ 2 & x > 2 \end{cases}$$

   (i) Determine $f_Y(y)$ in terms of $f_X(.)$, and (ii) Sketch $f_Y(y)$ when X is a uniform random variable over the interval [-3,3].

4. X and Y are random variables such that $Y = X^2$. Determine the pdf of Y if (a) X is Rayleigh, i.e.,

$$f_X(x) = \begin{cases} \frac{x}{\alpha} \exp\left(\frac{-x^2}{2\alpha}\right) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

   (b) X is a zero-mean Gaussian with variance σ^2, i.e., $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\dfrac{-x^2}{2\sigma^2}\right)$

5. If $Y = X^2$, show that $f_Y(y|X > 0) = \dfrac{1}{[1 - F_X(0)]2\sqrt{y}} f_X(\sqrt{y})$ for $y \geq 0$.

6. X is a uniform random variable over the interval [0,1]. Find a function $g(.)$ such that $Y = g(X)$ has the pdf given by

$$f_Y(y) = \frac{e^{-\sqrt{2}|y|}}{\sqrt{2}}.$$

7. Let X and Y be two independent random variables, where $f_X(x)$ is uniform over the interval [0,1] and $f_Y(y)$ is uniform over the interval [0,2]. Determine and sketch the pdf of the random variable $Z = X + Y$.

8. X and Y are two independent and identically distributed (i.i.d.) random variables with $f_X(x)$ given by

$$f_X(x) = \begin{cases} xe^{-x^2/2} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

   Define a new random variable $Z = X/Y$.

   (i) Determine $f_Z(z|Y = y)$, the conditional pdf of Z given $Y = y$.

(ii) Show that $f_Z(z) = \dfrac{2z}{(z^2+1)^2}$ for $z \geq 0$. If necessary, use $\displaystyle\int_0^\infty y^3 e^{-ky^2} dy = \frac{1}{2k^2}$ where k is a constant.

9. X and Y are independent and identically distributed (i.i.d.) Gaussian random variables with parameters $m = 0$ and σ^2. Let $R = \sqrt{(X^2+Y^2)}$ and $\Theta = \tan^{-1}(Y/X)$. Determine the joint pdf of R and Θ. Are R and Θ independent?

10. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$, where X and Y are arbitrary random variables. Express the joint pdf of Z and W in terms of the joint pdf of X and Y. If X and Y are i.i.d with

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

determine the joint pdf of Z and W.

11. Let X and Y be independent, jointly Gaussian zero-mean random variables with unit variance, i. e.,

$$f_{X,Y}(x, y) = \frac{1}{2\pi} \exp\left\{-\frac{x^2+y^2}{2}\right\}.$$

Define random variables Z and W as $Z = \sqrt{X^2+Y^2}$ and $W = X/Y$. Determine $f_{Z,W}(z, w)$, $f_Z(z)$, and $f_W(w)$.