EE5040 Problem Set 3

(1) $X = X_c + Z$ $\qquad X, \underline{Y}$ zero-mean

$\hat{X} = K_0 \underline{Y}$ where $K_0 R_Y = R_{XY}$

$$R_{XY} = E[X \underline{Y}^H] = E[(X_c + Z)\underline{Y}^H] = E[X_c \underline{Y}^H] + E[Z \underline{Y}^H]$$
$$= E[X_c \underline{Y}^H] = R_{X_c Y}.$$

Therefore, $K_0 R_Y = R_{X_c Y}$.

$$\hat{X_c} = K_0 \underline{Y} = \hat{X}.$$

[Even if $\underline{Y}$ was not zero mean, we will get the same result.

$$\hat{X} = K_0 (\underline{Y} - E[\underline{Y}])$$

where $K_0 R_Y = R_{XY}$ $\qquad R_Y = E[(\underline{Y} - E[\underline{Y}])(\underline{Y} - E[\underline{Y}])^H]$

$$R_{XY} = E[X_c(\underline{Y} - E[\underline{Y}])^H] = R_{X_c Y}.$$

$\Rightarrow \hat{X}_c = \hat{X}$]

(2) $\underline{\hat{X}} = K_0 \underline{Y}$ where K_0 is any solution to $K_0 R_Y = R_{XY}$.

We want to minimize $E[\underline{\tilde{X}}'^H W \underline{\tilde{X}}']$ for some $W \geq 0$. $(\underline{\tilde{X}}' = \underline{X} - \underline{\hat{X}}')$

$\underline{\hat{X}}' = K \underline{Y}$.

$$E[\underline{\tilde{X}}'^H W \underline{\tilde{X}}'] = E\left[(\underline{X} - K\underline{Y})^H W (\underline{X} - K\underline{Y})\right]$$
$$= E\left[(\underline{X} - K_0\underline{Y} + K_0\underline{Y} - K\underline{Y})^H W (\underline{X} - K_0\underline{Y} + K_0\underline{Y} - K\underline{Y})\right]$$
$$= E\left[(\underline{X} - K_0\underline{Y})^H W (\underline{X} - K_0\underline{Y})\right] + E\left[(\underline{X} - K_0\underline{Y})^H W (K_0\underline{Y} - K\underline{Y})\right] \quad (1)$$
$$+ E\left[(K_0\underline{Y} - K\underline{Y})^H W (\underline{X} - K_0\underline{Y})\right] + E\left[(K_0\underline{Y} - K\underline{Y})^H W (K_0\underline{Y} - K\underline{Y})\right]$$

We know that $K_0 \underline{Y}$ is the linear MMSE estimate of $\underline{X}$ from $\underline{Y}$.

Therefore, we know $\underbrace{(\underline{X} - K_0\underline{Y})}_{\text{error}} \perp \underline{Y}$, ie., $E[(\underline{X} - K_0\underline{Y})\underline{Y}^H] = 0$

Let $\underline{\tilde{X}} = \underline{X} - K_0\underline{Y}$. $\quad E[\underline{\tilde{X}}\underline{Y}^H] = 0 \Rightarrow E[\tilde{X}_i Y_j^*] = 0 \quad \forall i, j.$

This also implies $E[\tilde{x}_i^* y_j] = 0 \; \forall \; i, j$.

Now, consider the second term in eqn (1): Each entry is a linear combination of y_j's.

$$E\left[(\underline{X} - K_0 \underline{Y})^H W (K_0 \underline{Y} - K\underline{Y})\right] = E\left[\underline{\tilde{X}}^H W(K_0 - K)\underline{Y}\right]$$

$$= \text{linear combination of terms of the type } E[\tilde{x}_i^* y_j]$$

$$= 0.$$

Similarly, the third term is also 0.

$$\Rightarrow E\left[\underline{\tilde{X}}'^H W \underline{\tilde{X}}'\right] = \underbrace{E\left[(\underline{X} - K_0\underline{Y})^H W (\underline{X} - K_0 \underline{Y})\right]}_{\geq 0} + \underbrace{E\left[(K_0\underline{Y} - K\underline{Y})^H W (K_0\underline{Y} - K\underline{Y})\right]}_{\geq 0}.$$

$$\geq E\left[(\underline{X} - K_0 \underline{Y})^H W (\underline{X} - K_0 \underline{Y})\right]$$

($W \geq 0 \Rightarrow \underline{a}^H W \underline{a} \geq 0 \; \forall \underline{a}$)

This lower bound is achieved by $K = K_0$.

i.e. the linear MMSE estimate $K_0 \underline{Y}$ also minimizes $E\left[\underline{\tilde{X}}'^H W \underline{\tilde{X}}'\right]$ for any $W \geq 0$.

(3) $J(\underline{x}) = (\underline{x} - \underline{c})^H A (\underline{x} - \underline{c})$

Since A is Hermitian nonnegative-definite,

$$J(\underline{x}) \geq 0.$$

$\Rightarrow$ minimum possible value of $J(\underline{x})$ is 0.

For $\underline{x} = \underline{c} + \underline{d}$ where $A\underline{d} = \underline{0}$,

$$J(\underline{x}) = (\underline{c} + \underline{d} - \underline{c})^H A (\underline{c} + \underline{d} - \underline{c})$$

$$= \underline{d}^H A \underline{d} = \underline{d}^H \underline{0} = 0.$$

Thus $J(\underline{x}) = 0$ is achieved at $\underline{x} = \underline{c} + \underline{d}$ for any $\underline{d}$ satisfying $A\underline{d} = \underline{0}$.