EE5040: Adaptive Signal Processing
Problem Set 5: Steady-state performance analysis

1. (Sayed IV.29, Model with frequency offset) Consider data $\{X_i, \mathbf{Y}_i\}$ that satisfy the linear relation $X_i = \mathbf{K}_{0,i}^H \mathbf{Y}_i e^{-j\Omega i} + V_i$, where V_i denotes measurement noise and Ω models some constant frequency offset (Ω could be zero as well). Assume further that $\mathbf{K}_{0,i}$ varies according to the auto-regressive model: $\mathbf{K}_{0,i} = \mathbf{k}_0 + \boldsymbol{\theta}_i$ and $\boldsymbol{\theta}_i = \alpha\boldsymbol{\theta}_{i-1} + \mathbf{q}_i$ with $0 \le |\alpha| < 1$. In other words, $\mathbf{K}_{0,i}$ undergoes random variations around its mean $\mathbf{k}_0$, with the perturbations $\boldsymbol{\theta}_i$ being generated by a first-order auto-regressive model with a pole at α and a random inital condition denoted by $\boldsymbol{\theta}_{-1}$. Now, we will extend the analysis for the more general non-stationary data model described above.

   Consider adaptive filters of the form $\mathbf{K}_i = \mathbf{K}_{i-1} + \mu\mathbf{Y}_i g(E_i)$ with $\mathbf{K}_{-1}$ as the initial condition. Define the error quantities:

   $$\tilde{\mathbf{K}}_i = \mathbf{K}_{0,i} e^{j\Omega i} - \mathbf{K}_i$$

   $$E_{ai} = [\mathbf{K}_{0,i} e^{j\Omega i} - \mathbf{K}_{i-1}]^H \mathbf{Y}_i$$

   $$E_{pi} = [\mathbf{K}_{0,i} e^{j\Omega i} - \mathbf{K}_i]^H \mathbf{Y}_i$$

   (a) Establish the relations:

   $$E_{pi} = E_{ai} - \mu g^*(E_i)||\mathbf{Y}_i||^2$$

   and

   $$\tilde{\mathbf{K}}_i = \tilde{\mathbf{K}}_{i-1} - \mu\mathbf{Y}_i g(E_i) + \mathbf{c}_i e^{j\Omega(i-1)},$$

   where $\mathbf{c}_i = \mathbf{k}_0(e^{j\Omega} - 1) + \boldsymbol{\theta}_{i-1}(\alpha e^{j\Omega} - 1) + \mathbf{q}_i e^{j\Omega}$.

   (b) Establish the energy-conservation relation:

   $$||\tilde{\mathbf{K}}_i - \mathbf{c}_i e^{j\Omega(i-1)}||^2 + \bar{\mu}_i |E_{ai}|^2 = ||\tilde{\mathbf{K}}_{i-1}||^2 + \bar{\mu}_i |E_{pi}|^2$$

   where $\bar{\mu}_i = 1/||\mathbf{Y}_i||^2$ if $\mathbf{Y}_i \neq 0$ and $\bar{\mu}_i = 0$ otherwise.

   (c) Establish the following relations:

   $$\mathbf{K}_{0,i} = \mathbf{K}_{0,i-1} + \boldsymbol{\theta}_i - \boldsymbol{\theta}_{i-1}$$

   $$E_i = E_{ai} + V_i = \tilde{\mathbf{K}}_{i-1}^H \mathbf{Y}_i + (\mathbf{K}_{0,i} e^{j\Omega} - \mathbf{K}_{0,i-1})^H e^{-j\Omega(i-1)} \mathbf{Y}_\mathbf{i} + V_i$$

   (d) Show that this non-stationary model encompasses the model discussed in class.

2. (Sayed IV.30, Variance relation) Consider the same setting as in problem 1. Assume that $\mathbf{q}_i$ is an i.i.d. sequence with covariance matrix Q and independent of the initial conditions $\{\boldsymbol{\theta}_{-1}, \mathbf{K}_{-1}\}$, data $\{\mathbf{Y}_j\}$ for all j, and $\{X_j\}$ for all $j < i$. Assume further that the filter is operating in steady-state. By taking expectations of both sides in the energy-conservation relation, derive the variance relation. Show that it reduces to the relation derived in class if $\alpha = 1$ and $\Omega = 0$.