EE5040: Adaptive Signal Processing
Problem Set 4: Gradient Descent Method

1. (Sayed III.8, Prediction problem) A zero-mean stationary random process $\{U_i\}$ is generated by passing a zero-mean white sequence $\{V_i\}$ with variance σ_v^2 through a second-order auto-regressive model, namely, $U_i + \alpha U_{i-1} + \beta U_{i-2} = V_i$ for $i > -\infty$, where α and β are real numbers such that the roots of the characteristic equation $1 + \alpha z^{-1} + \beta z^{-2} = 0$ are strictly inside the unit circle. We wish to design a second-order predictor for the process $\{U_i\}$ of the form $\hat{U}_i = \mathbf{k}^H \mathbf{Y}$, where $\mathbf{Y} = [U_{i-1} \;\; U_{i-2}]^T$.

   (a) Verify that α and β must satisfy $|\beta| < 1$ and $|\alpha| < 1 + \beta$.

   (b) Find $\mathbf{R}_Y$ and $\mathbf{R}_{U_i Y}$. Establish that $(1-\beta)[(1+\beta)^2 - \alpha^2] > 0$.

   (c) Show that $\mathbf{k}_{opt} = [-\alpha \quad -\beta]^T$. Could you have guessed this answer more directly without evaluating it?

   (d) Verify that the eigenvalue spread of $\mathbf{R}_Y$ is $\rho = (\beta + 1 + |\alpha|)/(\beta + 1 - |\alpha|)$. Design a steepest descent algorithm that determines $\mathbf{k}_{opt}$ iteratively. Provide a condition on the step-size μ in terms of α and β in order to guarantee convergence.

   (e) Show that the value of the step-size that yields the fastest convergence, and the resulting time-constant, are

$$\mu^o = \frac{1-\beta}{1+\beta} \frac{(1+\beta)^2 - \alpha^2}{\sigma_v^2}, \quad \text{and} \quad \tau^o = \frac{1}{2\ln(|\alpha|/(\beta+1))}.$$

2. (Sayed III.4, Convergent step-size sequence) Consider the steepest descent algorithm with a time-variant step-size. Assume that μ_i converges to a positive value, say, $\mu_i \to \alpha > 0$ as $i \to \infty$. Show that if α satisfies $\alpha < 2/\lambda_{max}$, then $\mathbf{k}_i$ converges to $\mathbf{k}_{opt}$.

3. (Sayed III.5, Optimal step-size) Consider the optimal step-size in the iteration-dependent case of the steepest descent algorithm. Show that $1/\lambda_{max} \le \mu_i^o \le 1/\lambda_{min}$, where λ_{max} and λ_{min} denote the largest and smallest eigenvalues of $\mathbf{R}_Y$. Conclude that $\sum_{i=0}^{\infty} \mu_i^o$ diverges.

4. (Sayed III.3, Optimal step-size) Verify that the optimal iteration-dependent step-size is equivalent to the following expression:

$$\mu_i^o = \frac{\nabla_{\mathbf{k}_i} J(\mathbf{k}_i) \nabla_{\mathbf{k}_i}^H J(\mathbf{k}_i)}{\nabla_{\mathbf{k}_i} J(\mathbf{k}_i) \mathbf{R}_Y \nabla_{\mathbf{k}_i}^H J(\mathbf{k}_i)},$$

   in terms of the squared Euclidean norm of the gradient vector in the numerator, and the weighted squared Euclidean norm of the same vector in the denominator.