EC204: Networks & Systems
Solutions to Problem Set 6

1. (i) $x(t) = 2r(t) - 4r(t-2) + 2r(t-4)$

$$X(s) = \frac{2}{s^2} - \frac{4}{s^2}e^{-2s} + \frac{2}{s^2}e^{-4s} = \frac{2}{s^2}[1 - e^{-2s} + e^{-2s}]$$

(ii) Define $y_1(t)$ as

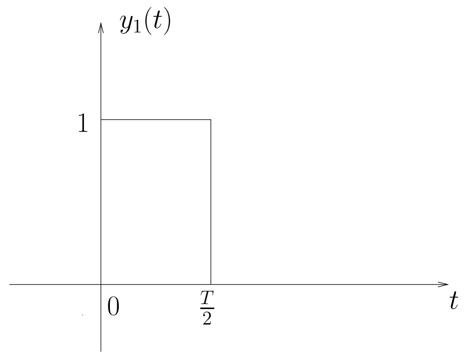

Figure 1: Problem 1

$$y_1(t) = u(t) - u\left(t - \frac{T}{2}\right)$$

$$\Longrightarrow Y_1(s) = \frac{1}{s} - \frac{1}{s}e^{-\frac{sT}{2}} = \frac{1}{s}(1 - e^{-\frac{sT}{2}})$$

$$\text{Now, } y(t) = \sum_{n=0}^{\infty} y_1(t - nT)$$

$$\Longrightarrow Y(s) = Y_1(s)[1 + e^{-sT} + e^{-2sT} + \ldots]$$

$$= \frac{Y_1(s)}{1 - e^{-sT}} = \frac{1}{s}\frac{(1 - e^{-sT/2})}{(1 - e^{-sT})}$$

$$= \frac{1}{s}\frac{1}{(1 + e^{-sT/2})} \quad \square$$

2. (a)

$$K_2 = \left.\frac{1}{(s+a)(s-p_1^*)}\right|_{s=p_1} = \frac{1}{(p_1+a)(p_1-p_1^*)}$$

$$K_3 = \left.\frac{1}{(s+a)(s-p_1)}\right|_{s=p_1^*} = \frac{1}{(p_1^*+a)(p_1^*-p_1)} = K_2^*$$

(b)

$$\begin{aligned}
K_2 &= \left.\frac{1}{(s+a)(s-p_1^*)(s-p_2)(s-p_2^*)}\right|_{s=p_1} \\
&= \frac{1}{(p_1+a)(p_1-p_1^*)(p_1-p_2)(p_1^*-p_2^*)} \\
K_3 &= \frac{1}{(p_1^*+a)(p_1^*-p_1)(p_1^*-p_2)(p_1^*-p_2^*)} = K_2^* \\
K_4 &= \frac{1}{(p_2+a)(p_2-p_1)(p_2-p_1^*)(p_2-p_2^*)} \\
K_5 &= \frac{1}{(p_2^*+a)(p_2^*-p_1)(p_2^*-p_1^*)(p_2^*-p_2)} = K_4^* \quad \square
\end{aligned}$$

3. (a)

$$\frac{s+2}{s^2+8s+15} = \frac{s+2}{(s+3)(s+5)} = \frac{-1/2}{s+3} + \frac{3/2}{s+5}$$

$$\Longrightarrow \mathcal{L}^{-1}\left[\frac{s+2}{s^2+8s+15}\right] = \left[\frac{-1}{2}e^{-3t} + \frac{3}{2}e^{-5t}\right]u(t)$$

(b)

$$\frac{s+1}{(s+2)^2(s+3)} = \frac{k_1}{s+3} + \frac{k_{21}}{s+2} + \frac{k_{22}}{(s+2)^2}$$

$$\begin{aligned}
k_1 &= \left.\frac{s+1}{(s+2)^2}\right|_{s=-3} = -2 \\
k_{22} &= \left.\frac{s+1}{s+3}\right|_{s=-2} = -1 \\
k_{21} &= \left.\frac{d}{ds}\left[\frac{s+1}{s+3}\right]\right|_{s=-2} = \left.\frac{(s+3)-(s+1)}{(s+3)^2}\right|_{s=-2} = 2
\end{aligned}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{s+1}{(s+2)^2(s+3)}\right] = \left[-2e^{-3t} + 2e^{-2t} - te^{-2t}\right]u(t)$$

(c)

$$\begin{aligned}\frac{2s^2+s+1}{s(s+2)} &= \frac{2s^2+s+1}{s^2+2s} \\ &= 2+\frac{(-3s+1)}{s^2+2s} \\ &= 2-\frac{(3s-1)}{s(s+2)} \\ &= 2-\left[\frac{k_1}{s}+\frac{k_2}{s+2}\right]\end{aligned}$$

$$\begin{aligned}k_1 &= \left.\frac{3s-1}{s+2}\right|_{s=0} = -\frac{1}{2} \\ k_2 &= \left.\frac{3s-1}{s}\right|_{s=-2} = \frac{7}{2}\end{aligned}$$

$$\Longrightarrow \mathcal{L}^{-1}\left[\frac{2s^2+s+1}{s(s+2)}\right] = 2\delta(t)+\frac{1}{2}e^{-t}u(t)-\frac{7}{2}e^{-2t}u(t)$$

(d)

$$\frac{2s+1}{(s+2)(s^2+1)^2} = \frac{2s+1}{(s+2)(s+\jmath)^2(s-\jmath)^2}$$

$$= \frac{k_1}{s+2}+\frac{k_{21}}{s+\jmath}+\frac{k_{22}}{(s+\jmath)^2}+\frac{k_{31}}{s-\jmath}+\frac{k_{32}}{(s-\jmath)^2}$$

$$\begin{aligned}k_1 &= \left.\frac{2s+1}{(s+\jmath)^2(s-\jmath)^2}\right|_{s=-2} = \left.\frac{2s+1}{(s^2+1)^2}\right|_{s=-2} = \frac{-3}{25} \\ k_{22} &= \left.\frac{2s+1}{(s+2)(s-\jmath)^2}\right|_{s=-\jmath} = \frac{-2\jmath+1}{(2-\jmath)(-4)} = \frac{-4+3\jmath}{20} \\ k_{32} &= \left.\frac{2s+1}{(s+2)(s+\jmath)^2}\right|_{s=\jmath} = \frac{2\jmath+1}{(2+\jmath)(-4)} = k_{22}^*\end{aligned}$$

$$\begin{aligned}k_{21} &= \left.\frac{(s+2)(s-\jmath)^2(2)-(2s+1)[(s+2)2(s-\jmath)+(s-\jmath)^2]}{(s+2)(s-\jmath)^2}\right|_{s=-\jmath} \\ &= \frac{2}{-2+\jmath} = \frac{-4-2j}{5}\end{aligned}$$

$k_{31} = k_{21}^*$

$$\mathcal{L}^{-1}\left[\frac{2s+1}{(s+2)(s^2+1)^2}\right] = k_1e^{-2t}u(t)+k_{21}e^{-jt}u(t)+k_{22}te^{-jt}u(t)+k_{31}e^{jt}u(t)+k_{32}te^{jt}u(t)$$

$$= k_1e^{-2t}u(t) + 2Re\left[k_{31}e^{jt}u(t) + k_{32}te^{jt}u(t)\right]$$

$$= \left[-\frac{3}{25}e^{-2t} + \left(-\frac{8}{5}\cos t - \frac{4}{5}\sin t\right) + \left(-\frac{2}{5}t\cos t + \frac{3}{10}t\sin t\right)\right]u(t).$$

(e)

$$\mathcal{L}^{-1}\left[\frac{1}{10^4s^2+10^2s+1}\right] = \mathcal{L}^{-1}\left[\frac{1}{\left(\frac{s}{10^{-2}}\right)^2+\left(\frac{s}{10^{-2}}\right)+1}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+s+1}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+\frac{1}{2})^2+\left(\frac{3}{2}\right)^2}\right]$$

$$= \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)e^{-t/2}u(t)$$

$$\Longrightarrow \mathcal{L}^{-1}\left[\frac{1}{10^4s^2+10^2s+1}\right] = (10^{-2})\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{200}t\right)e^{-t/200}u(t) \quad \square$$

4. $G(s) = \frac{1}{s+2}, \quad F(s) = \frac{1}{s} - \frac{1}{s}e^{-2s}$

$$F(s)G(s) = \frac{1}{s+2}(1-e^{-2s}) = \left[\frac{1/2}{s} + \frac{-1/2}{s+2}\right](1-e^{-2s})$$

$$= \frac{1}{2}[u(t) - e^{-2t}u(t)] - \frac{1}{2}[u(t-2) - e^{2(t-2)}u(t-2)]$$

$$= \frac{1}{2}[u(t) - u(t-2)] + \frac{1}{2}[e^{-2(t-2)}u(t-2) - e^{-2t}u(t)]$$

$$= \frac{1}{2}[1-e^{-2t}]u(t) + \frac{1}{2}[e^{-2(t-2)} - 1]u(t-2) \quad \square$$

5. (a)

$$v_C(0^+) = \lim_{s\to\infty} sV_C(s) = \lim_{s\to\infty}\frac{as^2+bs}{s^2+cs+d} = a$$

(b)

$$i(t) = C\frac{dv_C}{dt} \Longrightarrow I(s) = [sV_C(s) - v_C(0^+)]C$$
$$= \frac{s(as+b)}{s^2+cs+d} - a = \frac{(b-ac)s - ad}{s^2+cs+d}$$

(c)

$$i(0^+) = \lim_{s\to\infty} sI(s) = (b-ac)$$

(d)

$$v_L(t) = L\frac{di}{dt} \Longrightarrow V_L(s) = L[sI(s) - i(0^+)]$$
$$= [sI(s) - (b-ac)]L$$

(e)

$$v_L(0^+) = -ad - bc + ac^2 \quad \square$$

6.

$$E(s) = \frac{s+1}{(s+1)^2+4}$$

(a)

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s^2+s}{(s+1)^2+4} = 0$$

(b)

$$\int_0^\infty e(t)dt = E(0) = \frac{1}{5}$$

(c)

$$\int_0^\infty te(t)dt = -\left.\frac{dE(s)}{ds}\right|_{s=0} = -\left.\left\{\frac{(s+1)^2+4-2(s+1)(s+1)}{(s+1)^2+4}\right\}\right|_{s=0}$$

$$= -\left[\frac{5-2}{5}\right] = -\frac{3}{5} \quad \square$$