Cut-Set Bound

1 Upper Bounds on Relaying Rate

Though the unicast capacity of wireless networks is unknown, the upper bound on capacity is known. In [1], Cover *et al* defined the information theoretic cut-set bound which serves an upper bound for any multi-terminal communications. This is a natural extension of the Max-Flow Min-Cut theorem defined for multicommodity flow problems. In this section, we study the cut-set upper bounds for full-duplex relay networks [1] and half-duplex relay networks [2].

1.1 Full-duplex cut-set bound

Consider a relay network with m full-duplex wireless nodes defined by the graph $G = (V, E)$, where $V = \{1, 2, \ldots, m\}$ and $E \subseteq \{(u, v) : \forall\ u, v \in V\}$. Let X_i and Y_i be the random variables which denote the channel input and channel output at Node i. The network is defined by a probability distribution function $p\left(y_1, y_2, \ldots, y_m | x_1, x_2, \ldots x_m\right)$. Consider a unicast communication in this network from source S to destination D. Let $R_{S\to D}$ be the rate at which information is sent from source to destination. Let Ω and Ω^c be the partition of nodes in the network. The cut-set bound on the achievable rate $R_{S\to D}$ is defined in [1] and is reproduced here.

Theorem 1 *If the rate $R_{S\to D}$ is achievable, then there exists some probability distribution function $p\left(x_1, x_2, \ldots, x_m\right)$ such that*

$$R_{S\to D} \leq \min_{\Omega \subseteq V} I\left(X^{\Omega}; Y^{\Omega^c} | X^{\Omega^c}\right), \tag{1}$$

where $S \in \Omega$ and $D \in \Omega^c, X^{(\Omega)} = \{X_i : i \in \Omega\}$ and $Y^{\Omega^c} = \{Y_i : i \in \Omega^c\}$. The maximum information flow $R_{S\to D}$ across the cut-set edges is bounded by the conditional mutual information.

1.2 Half-duplex cut-set bound

A communication network could be in many states as opposed to the full-duplex networks where there is only one state. Examples of such networks include (i) a network of half-duplex nodes, (ii) wireless networks where channel state information of each link is modeled as finite Markov chains. M. Khojastepour *et al,* determined the cut-set bound for networks with many states in [2]. We concentrate on the cut-set bound in [2] as we are interested in designing relaying protocols for Gaussian half-duplex relay networks.

Suppose a half-duplex relay network operates in M states, $S_k = (I_k, J_k), 1 \leq k \leq M$, where I_k and J_k denote the nodes in transmit and receive mode in state S_k, respectively. Assume state S_k is active for a fraction of time λ_k. The half-duplex cut-set bound on the rate $R_{S\to D}$ defined in [2] is described in the following theorem.

Theorem 2 *Assume the sequence of states $S_k = (I_k, J_k), 1 \leq k \leq M$ is deterministic and known to all nodes in the network. If the rate $R_{S \to D}$ is achievable in a half-duplex network with M states, then there exists a probability distribution function $p(x_1, x_2, \ldots, x_m|k)$ such that*

$$R_{S \to D} \leq \sup_{\lambda_k, \sum \lambda_k = 1} \min_{\Omega \subseteq V} \sum_{k=1}^{M} \lambda_k I\left(X_k^{\Omega}; Y_k^{\Omega^c} | X_k^{\Omega^c}\right), \quad (2)$$

where $S \in \Omega$ and $D \in \Omega^c$, $X_k^{(\Omega)} = \{X_i : i \in \Omega \cap I_k\}$ and $Y_k^{(\Omega^c)} = \{Y_i : i \in \Omega^c \cap J_k\}$.

The information rate from source to destination in a multiterminal communications can be further maximized by considering a random sequence of states $S_1, S_2, \ldots, S_M$ as opposed to the deterministic sequence of states in Theorem 2. In [3], G. Kramer designed communication protocols which utilizes the half-duplex modes to increase the information rate in a relay channel. In our work, we restrict ourselves to the deterministic sequence of states as in [2, 4, 5, 6].

1.3 Computation of cut-set bound for Gaussian relay networks

Cut-set bounds described in (1) and (2) are maximized over the probability distribution functions. However, finding the optimal distribution is very difficult. Even if the optimal distribution is known, computing the conditional mutual information terms in (1) and (2) become cumbersome.

In this thesis, we are interested in Gaussian Relay Networks (GRNs). Let us assume Node i has an average power constraint P_i and a noise variance σ^2. In a Gaussian setting also, the exact cut-set bound is difficult to compute as the optimal distribution depends on the network topology and the desired communications. However, loosened cut-set bounds can be computed easily for GRNs. We describe such computations for full-duplex GRNs but the same applies to half-duplex GRNs. Let $\bar{C}_{FD}$ denote the full-duplex cut-set bound defined in (1) and is reproduced here as

$$\bar{C}_{FD} = \max_{p(x_1, x_2, \ldots, x_m)} \min_{\Omega \subseteq V} I\left(X^{\Omega}; Y^{\Omega^c} | X^{\Omega^c}\right), \quad (3)$$

where $S \in \Omega$ and $D \in \Omega^c$. Let us now interchange the maximum and minimum in (3). Thus, $\bar{C}_{FD}$ is upper bounded by

$$\bar{C}_{FD} \leq \min_{\Omega \subseteq V} \max_{p(x_1, x_2, \ldots, x_m)} I\left(X^{\Omega}; Y^{\Omega^c} | X^{\Omega^c}\right), \quad (4)$$

Consider any cut $\Omega \subseteq V$ in GRNs. The cut edges from Ω to Ω^c form a Mulitple Input Multiple Output (MIMO) channel. Therefore, the conditional mutual information $\max_{p(x_1, x_2, \ldots, x_m)} I\left(X^{\Omega}; Y^{\Omega^c} | X^{\Omega^c}\right)$ is equal to the capacity of the MIMO channel. This is achieved by picking $X_1, X_2, \ldots, X_m$ as jointly Gaussian random variables. Therefore, the full-duplex cut-set bound using the MIMO capacity is given by

$$\bar{C}_{FD} \leq \min_{\Omega \subseteq V} \max_{K_x^{\Omega}} C_{MIMO}(K_x^{\Omega}, H), \quad (5)$$

where $C_{MIMO}(K_x^\Omega, H) = \frac{1}{2}\log\det(\mathbb{I}_r + HK_x^\Omega H^*)$, $t = |\Omega|$, $r = |\Omega^c|$, $H = [h_{ij}], i \in J, j \in I$ is the channel matrix, h_{ij} is the channel gain from transmitter j to receiver i and K_x^Ω is the $t \times t$ covariance matrix of the random variables in X^Ω which satisfies the power constraint

$$\text{trace}(K_x^\Omega) \leq P_{\text{tot}}^\Omega = \sum_{i\in\Omega} P_i.$$

The covariance matrix K_x^Ω which maximizes the MIMO capacity in (5) is computed by solving a semi-definite programming [7]. Since K_x^Ω is a positive semi-definite matrix with trace constraint, $P_{\text{tot}}^\Omega \mathbb{I}_t - K_x$ is also a positive semi-definite matrix [8]. Also, $\log\det(.)$ is an increasing function on the cone of positive semi-definite matrices. As $P_{\text{tot}}^\Omega \mathbb{I}_t - K_x^\Omega$ is a positive semi-definite matrix, the inequality

$$\log\det(P_{\text{tot}}^\Omega \mathbb{I}_t) \geq \log\det K_x^\Omega, \tag{6}$$

holds for every Ω. Therefore, we replace K_x^Ω by P_{tot}^Ω in (5). Thus, the cut-set bound becomes

$$\bar{C}_{FD} \leq \min_{\Omega\subseteq V} C_{MIMO}(P_{\text{tot}}^\Omega \mathbb{I}_t, H). \tag{7}$$

We call the cut-set bound determined using (7) as the *Total Power Constraint Full-Duplex (TPC-FD) cut-set bound*. The MIMO capacity in (7) is simply a function of the total transmit power P_{tot}^Ω and the channel matrix H. Hence, $\bar{C}_{FD}$ can be evaluated easily without optimizing the covariance matrix K_x^Ω.

Let $P_{\max}^\Omega = \max_{i\in\Omega} P_i$ be the maximum transmit power in the cut Ω. Therefore, $P_{\text{tot}}^\Omega \leq tP_{\max}^\Omega$. The bound in (7) thus becomes

$$\begin{aligned}
\bar{C}_{FD} &\leq \min_{\Omega\subseteq V} \frac{1}{2}\log\det(\mathbb{I}_r + tP_{\max}^\Omega HH^*), \\
&\leq \min_{\Omega\subseteq V} \frac{1}{2}\log\det(t\mathbb{I}_r + tP_{\max}^\Omega HH^*), \\
&\leq \min_{\Omega\subseteq V} \frac{r}{2}\log t + \frac{1}{2}\log\det(\mathbb{I}_r + P_{\max}^\Omega HH^*).
\end{aligned} \tag{8}$$

This implies the TPC-FD cut-set bound grows linearly with the network size and often becomes loose for larger network size. However, using the above approach one can determine a quick upper bound for any network topology.

The half-duplex cut-set bound for Gaussian relay networks can also be computed in the same way by replacing the conditional mutual information with the MIMO capacity. Let $\bar{C}_{HD}$ denote the half-duplex cut-set bound. Using the matrix inequality in (6), $\bar{C}_{HD}$ can be written as

$$\bar{C}_{HD} \leq \sup_{\lambda_k, \sum\lambda_k=1} \min_{\Omega\subseteq V} \sum_{k=1}^{M} \lambda_k C_{MIMO}(P_{\text{tot}}^k \mathbb{I}_q, H_k), \tag{9}$$

where $p = |\Omega^c \cap J_k|$, $q = |\Omega \cap I_k|$, $P_{\text{tot}}^k = \sum_{i\in\Omega\cap I_k} P_i$ and H_k is the $p \times q$ channel submatrix. We call the cut-set bound determined using (9) as the *Total Power Constraint Half-Duplex*

(TPC-HD) cut-set bound. For a given channel conditions and transmit power constraint, $\bar{C}_{HD}$ is maximized by optimal choosing the time sharing variables $\lambda_1, \lambda_2, \ldots, \lambda_M$. The optimization in (9) is a linear program and can be solved efficiently for $\lambda_1, \lambda_2, \ldots, \lambda_M$ [7].

1.3.1 Illustration of TPC-HD bound:

We consider the diamond channel to illustrate the computation of total power constraint half-duplex cut-set bound. The cut-set upper bound $\bar{C}_{HD}$ defined in (9) is computed by solving the following linear optimization problem.

$$\max \bar{C}_{HD} \tag{10}$$

subject to

$$\begin{aligned}
\bar{C}_{HD} &\leq \lambda_1 C\left(\frac{(h_{12}^2+h_{13}^2)P}{\sigma^2}\right) + \lambda_2 C\left(\frac{h_{13}^2 P}{\sigma^2}\right) + \lambda_3 C\left(\frac{h_{12}^2 P}{\sigma^2}\right),\\
\bar{C}_{HD} &\leq \lambda_1 C\left(\frac{h_{13}^2 P}{\sigma^2}\right) + \lambda_2\left(C\left(\frac{h_{13}^2 P}{\sigma^2}\right) + C\left(\frac{h_{24}^2 P}{\sigma^2}\right)\right) + \lambda_4 C\left(\frac{h_{24}^2 P}{\sigma^2}\right),\\
\bar{C}_{HD} &\leq \lambda_1 C\left(\frac{h_{12}^2 P}{\sigma^2}\right) + \lambda_3\left(C\left(\frac{h_{12}^2 P}{\sigma^2}\right) + C\left(\frac{h_{34}^2 P}{\sigma^2}\right)\right) + \lambda_4 C\left(\frac{h_{34}^2 P}{\sigma^2}\right),\\
\bar{C}_{HD} &\leq \lambda_2 C\left(\frac{h_{24}^2 P}{\sigma^2}\right) + \lambda_3 C\left(\frac{h_{34}^2 P}{\sigma^2}\right) + \lambda_4 C\left(\frac{(h_{24}+h_{34})^2 P}{\sigma^2}\right),\\
\sum_{k=1}^{M}\lambda_k &= 1, \lambda_k \geq 0.
\end{aligned}$$

An upper bound on $\bar{C}_{HD}$ is found by solving the dual problem of (10) where we rely on the fact that every feasible solution of the dual problem gives an upper bound on the primal problem. Since the network is symmetric, the primal and dual programs have the same form. The dual optimization problem is

$$\min \bar{C}_{HD}, \tag{11}$$

subject to

$$\begin{aligned}
\bar{C}_{HD} &\geq \tau_1 C\left(\frac{(h_{12}^2+h_{13}^2)P}{\sigma^2}\right) + \tau_2 C\left(\frac{h_{13}^2 P}{\sigma^2}\right) + \tau_3 C\left(\frac{h_{12}^2 P}{\sigma^2}\right),\\
\bar{C}_{HD} &\geq \tau_1 C\left(\frac{h_{13}^2 P}{\sigma^2}\right) + \tau_2\left(C\left(\frac{h_{13}^2 P}{\sigma^2}\right) + C\left(\frac{h_{24}^2 P}{\sigma^2}\right)\right) + \tau_4 C\left(\frac{h_{24}^2 P}{\sigma^2}\right),\\
\bar{C}_{HD} &\geq \tau_1 C\left(\frac{h_{12}^2 P}{\sigma^2}\right) + \tau_3\left(C\left(\frac{h_{12}^2 P}{\sigma^2}\right) + C\left(\frac{h_{34}^2 P}{\sigma^2}\right)\right) + \tau_4 C\left(\frac{h_{34}^2 P}{\sigma^2}\right),\\
\bar{C}_{HD} &\geq \tau_2 C\left(\frac{h_{24}^2 P}{\sigma^2}\right) + \tau_3 C\left(\frac{h_{34}^2 P}{\sigma^2}\right) + \tau_4 C\left(\frac{(h_{24}+h_{34})^2 P}{\sigma^2}\right),\\
\sum_{k=1}^{M}\tau_k &= 1, \tau_k \geq 0.
\end{aligned}$$

The solution of (11) gives an upper bound on $\bar{C}_{HD}$. However, this solution is not tight always. In [4], closed form expressions are determined by solving (11) for all values of Δ. The achievable rates under the MDF-BC and MDF-MAC protocol are also derived and compared with the upper bound. The performance of both protocols are atmost 0.71 bits away from capacity.

1.4 Tight upper bounds for Gaussian relay networks

The cut-set bounds in (7) and (9) are easier to compute using the covariance matrix inequality in (6). However, these bounds are not tight for large networks as noticed in (8). Also, these bounds use a total power of tP_{tot}^{Ω} whereas the total power available is only P_{tot}^{Ω} for a cut Ω of size t. Let us now derive tight upper bounds for GRNs in this subsection. In the new upper bounds, we determine the MIMO capacity with strict power constraints, namely, sum power constraint and per-user power constraints.

In a MIMO channel with sum power constraint, the capacity is achieved by diagonlizing the channel. This is done by transmitter precoding, waterfilling power allocation along the eigen channels and receiver post processing [9]. The exact MIMO capacity of a $t \times r$ MIMO channel with channel matrix H and sum power constraint P_{tot} is given by

$$C_{MIMO}^{wf}(P_{\text{tot}}, H) = \sum_{i=1}^{n_{min}} \frac{1}{2} \log\left(1 + \frac{P_i^* \lambda_i^2}{\sigma^2}\right), \tag{12}$$

where n_{min} is the number of non-zero singular values of H, $\lambda_1, \lambda_2, \ldots, \lambda_{n_{min}}$ are the singular values of H and $P_1^*, P_2^*, \ldots, P_{n_{min}}^*$ are the waterfilling power allocation levels such that

$$\begin{aligned} P_i^* &= \left(\mu - \frac{\sigma^2}{\lambda_i^2}\right)^+, \\ \sum_i P_i^* &= P_{\text{tot}}. \end{aligned}$$

Each cut Ω in a Gaussian multi-terminal network is a MIMO channel and the available sum power P_{tot}^{Ω}. To determine a tight upper bound, we utilize the MIMO capacity in (12). Thus, the conditional mutual information of each cut is replaced by $C_{MIMO}^{wf}(P_{\text{tot}}, H)$, where H is the channel matrix of cut edges. The tight cut-set upper bound with sum power constraint is given by

$$\bar{C}_{FD} \leq \min_{\Omega \subseteq V} \; C_{MIMO}^{wf}(P_{\text{tot}}^{\Omega}, H). \tag{13}$$

The upper bound in (13) is called as *WaterFilling Full-Duplex (WF-FD) cut-set upper bound.* As the sum power constraint is not violated, WF-FD bound is always tighter than TPC-FD bound. However, waterfilling policy is not practical as it assumes transmitters can share the power to maximize the MIMO capacity.

A tighter cut-set bound for GRNs can be obtained by evaluating the MIMO capacity with per user power constraint rather than the sum power constraint. The MIMO sum capacity with per user power constraint is obtained by solving the following optimization problem.

$$\max_{K_{x,pu}^{\Omega}} \quad C_{MIMO}^{pu}(K_{x,pu}^{\Omega}, H) \tag{14}$$

subject to

$$\begin{aligned}
C_{MIMO}^{pu}(K_{x,pu}^{\Omega}, H) &= \frac{1}{2}\log\det(\mathbb{I}_r + HK_{x,pu}^{\Omega}H^*),\\
K_{x,pu}^{\Omega} &\succeq 0,\\
\operatorname{diag}(K_{x,pu}^{\Omega}) &\preceq \{P_i : i \in \Omega\}.
\end{aligned}$$

The notation $\succeq$ indicates the covariance matrix $K_{x,pu}^{\Omega}$ is a positive semi-definite matrix and $\preceq$ denotes the elementwise inequality. The above optimization is a semi-definite programming which searches for $K_{x,pu}^{\Omega}$ over the cone of positive semi-definite matrices [7]. The tight cut-set upper bound using per user power constraint is given by

$$\bar{C}_{FD} \leq \min_{\Omega \subseteq V} \quad C_{MIMO}^{pu}(K_{x,pu}^{\Omega}, H). \tag{15}$$

The bound in (15) is called as *Per-User Power Constraint Full-Duplex (PUPC-FD) cut-set bound*. This bound is tighter than the TPC-FD and WF-FD bounds as the MIMO capacity is always smaller with per user power constraints.

In all the upper bounds for GRNs discussed so far in this section, we computed the conditional mutual information of each cut as the MIMO capacity. The cut capacity is maximized by the Gaussian distribution, $\mathcal{N}(0, K_x^{\Omega})$. The covariance matrix K_x^{Ω} is optimized for each cut independently. But, the transmit random variables X^{Ω} are not disjoint for each cut, Ω. The correlation among the random variables is not consistent across the cuts for each choice of K_x^{Ω}. For example, in the diamond channel, the cut $\Omega_1 = \{1, 2\}$ is maximized when the random variables X_2 and X_3 are independent whereas the cut $\Omega_2 = \{1, 2, 3\}$ is maximized when $X_2 = X_3$, *i.e.,* X_2 and X_3 are fully correlated. So, we can tighten the upper bound by making the correlation among the random variables consistent across the cuts. Let K_x be the $m \times m$ covariance matrix of the transmit random variables $X_1, X_2, \ldots, X_m$ present in the network. Let $K_{x,corr}^{\Omega}$ be the conditional covariance matrix of the random variables in X^{Ω} given the random variables in X^{Ω^c}. For Gaussian random variables, $K_{x,corr}^{\Omega}$ is computed from K_x^{Ω} using Schur complement. Let Σ_{11}, Σ_{22} be the covariance matrices of the random variables in X^{Ω} and X^{Ω^c}, respectively. Let Σ_{12}, Σ_{21} be the cross covariance matrices of the random variables in X^{Ω} and X^{Ω^c}. Therefore, K_x^{Ω} is written as

$$K_x^{\Omega} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

The conditional covariance matrix $K_{x,corr}^{\Omega}$ is thus given by

$$K_{x,corr}^{\Omega} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

Using the correlation, the cut-set upper bound of Gaussian multi-terminal networks is

Bound	Expression	Optimization type
FD	$\bar{C}_{FD} = \max_{p(x_1,x_2,\ldots,x_m)} \min_{\Omega\subseteq V} I\left(X^{\Omega}; Y^{\Omega^c} \vert X^{\Omega^c}\right)$	-
TPC-FD	$\bar{C}_{FD} \leq \min_{\Omega\subseteq V} C_{MIMO}(P^{\Omega}_{\text{tot}}\mathbb{I}_t, H)$	Linear
WF-FD	$\bar{C}_{FD} \leq \min_{\Omega\subseteq V} C^{wf}_{MIMO}(P^{\Omega}_{\text{tot}}, H)$	Non-linear
PUPC-FD	$\bar{C}_{FD} \leq \min_{\Omega\subseteq V} C^{pu}_{MIMO}(K^{\Omega}_{x,pu}, H)$	SDP
C-FD	$\bar{C}_{FD} \leq \min_{\Omega\subseteq V} C^{pu}_{MIMO}(K^{\Omega}_{x,corr}, H)$	SDP

Table 1: Cut-set bound expressions for full-duplex GRNs

computed as

$$\begin{aligned}
\bar{C}_{FD} &\leq \min_{\Omega\subseteq V} \max_{\mathcal{N}(0,K_x)} I\left(X^{\Omega}; Y^{\Omega^c} \vert X^{\Omega^c}\right), \\
&\leq \min_{\Omega\subseteq V} C^{pu}_{MIMO}(K^{\Omega}_{x,corr}, H). \qquad (16)
\end{aligned}$$

The *Correlation Full-Duplex (C-FD) cut-set bound* in (16) is the tightest bound described so far. However, computing the C-FD bound is difficult as the network is optimized altogether instead of optimizing the individual cuts. For large network sizes, evaluating C-FD bound is cumbersome.

The correlation bound extends naturally for half-duplex networks. In a half-duplex network, the cut-set bound is maximized over the probability distribution function $p\left(x_1, x_2, \ldots, x_m \vert k\right)$ which is defined for each state S_k. To compute the losened cut-set bound, we chose $p\left(x_1, x_2, \ldots, x_m \vert k\right)$ as $\mathcal{N}(0, K^{k,\Omega}_x)$ for each state S_k and cut Ω. As in the full duplex cut-set bound, we utilize correlation to tighten the cut-set bound for half-duplex networks. The correlation bound for half-duplex networks is defined as

$$\begin{aligned}
\bar{C}_{HD} &\leq \sup_{\lambda_k} \min_{\Omega\subseteq V} \max_{p(x_1,x_2,\ldots,x_m\vert k)} \sum_{k=1}^{M} \lambda_k I\left(X^{\Omega}_k; Y^{\Omega^c}_k \vert X^{\Omega^c}_k\right), \\
&\leq \sup_{\lambda_k} \min_{\Omega\subseteq V} \max_{\mathcal{N}(0,K^k_x)} \sum_{k=1}^{M} \lambda_k I\left(X^{\Omega}_k; Y^{\Omega^c}_k \vert X^{\Omega^c}_k\right), \\
&\leq \sup_{\lambda_k} \min_{\Omega\subseteq V} \max_{\mathcal{N}(0,K^k_x)} \sum_{k=1}^{M} \lambda_k C^{pu}_{MIMO}(K^{k,\Omega}_{x,corr}, H_k), \qquad (17)
\end{aligned}$$

where K^k_x is the optimal covariance matrix for state S_k and $K^{k,\Omega}_{x,corr}$ is the conditional covariance matrix of the random variables in X^{Ω}_k given $X^{\Omega^c}_k$. Tables 1 and 2 summarize all the bounds for full-duplex GRNs and half-GRNs discussed in this section for quick reference.

1.5 Numerical Evaluation

The diamond channel wit half-duplex relays is considered for numerical evaluation of the cut-set bounds. The channel conditions are set as follows: (i) channel condition A:

Bound	Expression	Optimization type
HD	$\bar{C}_{HD} = \sup_{\lambda_k} \max_{p(x_1,\ldots,x_m\|k)} \min_{\Omega\subseteq V} \sum_{k=1}^{M} \lambda_k I\left(X_k^{\Omega}; Y_k^{\Omega^c} \| X_k^{\Omega^c}\right)$	-
TPC-HD	$\bar{C}_{FD} \leq \sup_{\lambda_k} \min_{\Omega\subseteq V} \sum_{k=1}^{M} \lambda_k C_{MIMO}(P_{\text{tot}}^k \mathbb{I}_q, H_k)$	Linear
WF-HD	$\bar{C}_{FD} \leq \sup_{\lambda_k} \min_{\Omega\subseteq V} \sum_{k=1}^{M} \lambda_k C_{MIMO}^{wf}(P_{\text{tot}}^k, H_k)$	Non-linear
PUPC-HD	$\bar{C}_{FD} \leq \sup_{\lambda_k} \min_{\Omega\subseteq V} \sum_{k=1}^{M} \lambda_k C_{MIMO}^{pu}(K_{x,pu}^{k,\Omega}, H_k)$	SDP
C-HD	$\bar{C}_{FD} \leq \sup_{\lambda_k} \min_{\Omega\subseteq V} \sum_{k=1}^{M} \lambda_k C_{MIMO}^{pu}(K_{x,corr}^{k,\Omega}, H_k)$	SDP

Table 2: Cut-set bound expressions for half-duplex GRNs

$h_{12} = h_{13} = \alpha$, $h_{24} = 0.25$ and $h_{34} = 1$, and (ii) channel condition B: $h_{12} = h_{13} = \alpha$, $h_{24} = 1$ and $h_{34} = 1$.

Channel condition A: Fig. 1 shows the performance of the MDF-BC and MDF-MAC protocols and comparison of the cut-set bounds defined in the previous subsection under channel condition A. The MDF-MAC protocol is used to find the achievabe rate when $\Delta > 0$ ($\alpha \geq -3$dB) and the MDF-BC protocol is used when $\Delta < 0$ ($\alpha \leq -3$ dB). We notice that the bounds are in the increasing order for all values of α, *i.e.*, $TPC-HD \geq WF-HD \geq PUPC-HD \geq C-HD$ as described in the previous section. For $\alpha \leq -5.57$ dB, the cut $\Omega = \{1\}$ dominates. Since there is only one transmitter in this cut, all four cut-set bounds yield the same value. For $-5.57 \leq \alpha \leq 1.01$ dB, the cut $\Omega = \{1, 2\}$ dominates. We observe the cut-set bound obtained using different methods are slightly different. The methods show significant difference when $\alpha \geq 1.01$ dB where the sink cut $\Omega = \{1, 2, 3\}$ is dominant. The correlation bound is tight because the random variabes X_2 and X_3 are correlated. PUPC-HD cut-set bound allows full cooperation across the transmitters in State S_4 to maximize the cut capacity. However, the cut-capacity in States S_2 and S_3 are maximized by choosing X_2 and X_3 independent. Because of this inconsistency, PUPC-HD bound is looser than the C-HD bound. TPC-HD bound and WF-HD bound allocate all the power along the eigen channel in State S_4 to maximize the min-cut capacity. This power allocation policy is very impractical which makes those bound looser. The achievable rate of the MDF protocol is very close to the correlation bound.

Channel condition B: Fig. 2 shows the comparison under channel condition B. When $\Delta \leq 0$ ($\alpha < 0$ dB), the cut-set bound is determined by the cut $\Omega = \{1\}$. So, all the four methods yield the same value as there is no power violation with one transmitter. When $\Delta > 0$ ($\alpha \geq 0$ dB), the cut-set bound is determined by the sink cut $\Omega = \{1, 2, 3\}$. Since there is only eigen channel and $h_{24} = h_{34}$ in State S_4, the PUPC-HD bound, WF-HD bound and TPC-HD bound yield the same value. The correlation bound is tight as

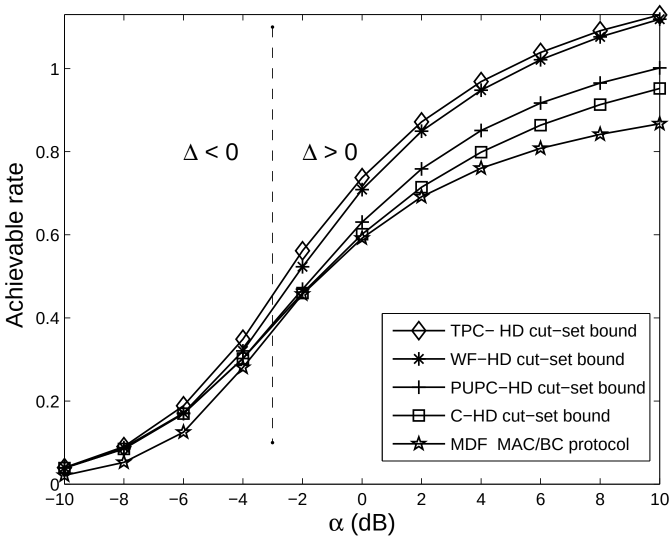

Figure 1: Performance of the MDF protocol in the diamond channel, Channel condition A

it optimizes the network altogether which makes the correlation consistent among the states and cuts. The MDF protocol achieves capacity for $\Delta = 0$ and is very close to the cut-set bound for all vaues of α.

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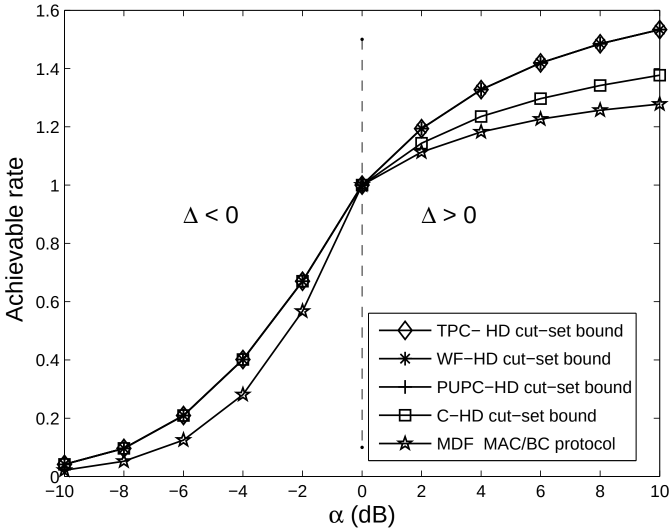

Figure 2: Performance of the MDF protocol in the diamond channel, Channel condition B

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